A canonical model for presheaf semantics

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Abstract

Presheaf models [Osius, 1975, Fourman and Scott, 1979, Moerdijk and Mac Lane, 1992] are a simple generalization of Kripke models and provide one of the most natural semantics for intuitionistic predicate logic. Although a completeness result for this semantics is known, this is usually obtained either non-constructively, by defining a canonical Kripke model, or indirectly, e.g. via categorical equivalences with $\Omega-$models as in [Troelstra and Van Dalen, 1988]. This paper describes an elementary canonical model construction by means of which completeness is established in a perspicuous, direct and constructive way.

Presheaf semantics. We start by recalling the definition of presheaf semantics for intuitionistic logic.

Definition 1 (Presheaf models for intuitionistic logic)**.** Let $\mathcal{L}$ be a predicate logic language with equality. A *presheaf model* for $\mathcal{L}$ consist of a presheaf of first-order structures for $\mathcal{L}$ over a Grothendieck site $(\mathcal{C}, \lhd)$. That is, to any object u of $\mathcal{C}$ we associate an $\mathcal{L}-$structure M_u and to any $f : v \to u$ we associate a homomorphism $\upharpoonright_f : M_u \to M_v$ in a functorial way.

The domain of M_u is denoted $|M_u|$, while the interpretation of a function symbol f and of a relation symbol R in M_u are denoted f_u, R_u respectively.

In addition, we require our structure to satisfy separateness and an analogous property asserting the local character of atomic formulas.

Separateness For any elements a, b of M_u, if $u \lhd \{f_i : u_i \to u \mid i \in \mathcal{I}\}$ and for any $i \in \mathcal{I}$ we have $a\upharpoonright_{f_i} = b\upharpoonright_{f_i}$, then $a = b$.

Local character of atoms For any $n-$ary relation symbol R and any tuple $(a_1, \ldots, a_n)$ from M_u, if $u \lhd \{f_i : u_i \to u \mid i \in \mathcal{I}\}$ and for any $i \in \mathcal{I}$ we have $(a_1\upharpoonright_{f_i}, \ldots, a_n\upharpoonright_{f_i}) \in R_{u_i}$, then $(a_1, \ldots, a_n) \in R_u$.

Any assignment ν into a structure $|M|$ extends in the usual way to an interpretation $[t]_\nu$ of any term t into M. Moreover, if in a presheaf model we have $f : v \to u$ and an assignment ν into M_u, then an assignment $\upharpoonright_f \circ \nu$ into M_v is naturally induced. We will allow ourselves to write again ν for the resulting assignment.

Definition 2 (Forcing on a presheaf model)**.** Formulas of $\mathcal{L}$ can be interpreted on an object u of a given presheaf model M, relative to an assignment ν into M_u as follows.

1. $u \Vdash_\nu R(t_1,\ldots,t_n) \iff ([t_1]_\nu,\ldots,[t_n]_\nu) \in R_u$.
2. $u \Vdash_\nu t_1 = t_2 \iff [t_1]_\nu = [t_2]_\nu$.
3. $u \Vdash_\nu \varphi \wedge \psi \iff u \Vdash_\nu \varphi$ and $u \Vdash_\nu \psi$.
4. $u \Vdash_\nu \varphi \vee \psi \iff$ there is a covering family $\{f_i : u_i \to u \,|\, i \in \mathcal{I}\}$ such that for any $i \in \mathcal{I}$ we have either $u_i \Vdash_\nu \varphi$ or $u_i \Vdash_\nu \psi$.
5. $u \Vdash_\nu \bot \iff u \lhd \emptyset$.
6. $u \Vdash \varphi \to \psi \iff$ for any $f : v \to u$, if $v \Vdash \varphi$ then $v \Vdash \psi$.
7. $u \Vdash_\nu \forall x\varphi \iff$ for all $f : v \to u$ and all $a \in M_v$ we have $v \Vdash_{\nu[x\mapsto a]} \varphi$.
8. $u \Vdash_\nu \exists x\varphi \iff$ there exist a covering family $\{f_i : u_i \to u \,|\, i \in \mathcal{I}\}$ and elements $a_i \in |M_{u_i}|$ for $i \in \mathcal{I}$ such that $u_i \Vdash_{\nu[x\mapsto a_i]} \varphi$ for any index i.

Notice that the usual Kripke semantics is obtained as a particular case when the underlying Grothendieck site is a poset equipped with the trivial covering $u \lhd \mathcal{F} \iff u \in \mathcal{F}$.

Definition 3 (Entailment)**.** We say that Γ *entails* φ in presheaf semantics, in symbols $\Gamma \models_{ps} \varphi$, in case for any object u of any presheaf model and for any assignment ν into M_u, if $u \Vdash_\nu \gamma$ for any $\gamma \in \Gamma$ then $u \Vdash_\nu \varphi$.

Intuitionistic logic is easily seen to be sound w.r.t. this semantics: if $\Gamma \vdash \varphi$ then $\Gamma \models_{ps} \varphi$. It is also known to be complete. In the literature, this is established in two ways (see Troelstra and Van Dalen [1988]): either by constructing a canonical *Kripke* model, which requires the use of Henkin constants and involves non-constructive steps; or indirectly, making use of a categorical equivalence result with $\Omega-$valued structures: turning this into a direct proof is possible, but as an algebraic completion is needed to obtain a canonical $\Omega-$valued structure, the result is not very natural.

My purpose here is to point out an elementary construction, simpler than both the previous ones, by means of which completeness can be established a direct and constructive way.

Construction of the canonical presheaf model. Let Γ be a fixed set of sentences. We denote by $\overline{\varphi}$ the equivalence class of the formula φ under the relation $\equiv_\Gamma$ of logical equivalence modulo Γ: $\varphi \equiv_\Gamma \psi \iff \Gamma \vdash \varphi \leftrightarrow \psi$. If Δ is a set of formulae, we put $\overline{\Delta} = \{\overline{\delta} \,|\, \delta \in \Delta\}$. We order the set $\overline{\mathcal{L}}$ of such equivalence classes according to entailment as usual: $\overline{\varphi} \le \overline{\psi}$ iff $\Gamma \vdash \varphi \to \psi$.

It is a well-known fact that the resulting poset $\mathbb{L}_\Gamma$ is a Heyting algebra. Heyting algebras are equipped with a natural structure of Grothendieck site when covering is defined by $a \lhd \{a_i \,|\, i \in \mathcal{I}\} \iff a$ is the least upper bound of the family $\{a_i \,|\, i \in \mathcal{I}\}$. In the particular case of our structure, this definition boils down to:

$$\overline{\varphi} \lhd \{\overline{\varphi_i} \,|\, i \in \mathcal{I}\} \quad \iff \quad \text{for all } \chi, \quad \Gamma, \varphi \vdash \chi \quad \text{iff} \quad (\Gamma, \varphi_i \vdash \chi \text{ for all } i \in \mathcal{I})$$

Thus, $\overline{\varphi} \lhd \{\overline{\varphi_i} \mid i \in \mathcal{I}\}$ amounts to Γ, φ proving a formula if and only if this can be proven from any Γ, φ_i. Moreover, the "only if" direction is trivial, since the elements of a covering family are required to be subobjects.

We now turn to the definition of a presheaf over this site. If φ is a formula and t, t' are terms, we write $t \equiv^{\varphi} t'$ in case $\Gamma, \varphi \vdash t = t'$. We denote by t^{φ} the equivalence class of t under $\equiv^{\varphi}$.

Definition 4. The *canonical presheaf model* $\mathcal{K}_\Gamma$ is the presheaf of first-order structures over the canonical site $(\mathbb{L}_\Gamma, \lhd)$ defined as follows.

1. To any object $\overline{\varphi}$ we associate the first-order structure $M_{\overline{\varphi}}$ such that:
   - its domain $|M_{\overline{\varphi}}|$ is the set of equivalence classes t^{φ} of terms;
   - an $n-$ary function symbol f is interpreted as the function $f_{\overline{\varphi}}$ mapping a tuple $(t_1^{\varphi}, \ldots, t_n^{\varphi})$ to $f(t_1, \ldots, t_n)^{\varphi}$; in particular, the interpretation of constants is $c_{\overline{\varphi}} = c^{\varphi}$;
   - an $n-$ary relation symbol R is interpreted as the $n-$ary relation $R_{\overline{\varphi}}$ given by $(t_1^{\varphi}, \ldots, t_n^{\varphi}) \in R_{\overline{\varphi}} \Longleftrightarrow \Gamma, \varphi \vdash R(t_1, \ldots, t_n)$.
2. If $\overline{\varphi} \leq \overline{\psi}$, the restriction map $\upharpoonright_{\overline{\varphi}} : M_{\overline{\psi}} \to M_{\overline{\varphi}}$ is defined by $t^{\psi} \upharpoonright_{\overline{\varphi}} = t^{\varphi}$.

It is straightforward to verify that $\mathcal{K}_\Gamma$ is well-defined and that it is indeed a presheaf model. Incidentally, we remark that $\mathcal{K}_\Gamma$ is not a sheaf, i.e. not every compatible family admits a gluing: one can find simple counterexamples, but we lack the space to present one.

On any point $\overline{\varphi}$ we have a canonical valuation ν_K given by $\nu_K(x) = x^{\varphi}$. It is straightforward to show that this extends to $[t]_{\nu_K} = t^{\varphi}$ for all terms t. As a consequence we have the following simple fact.

Remark 5. For any formula $\psi(x)$ and term t, $\overline{\varphi} \Vdash_{\nu_K[x \mapsto t^{\varphi}]} \psi(x)$ iff $\overline{\varphi} \Vdash_{\nu_K} \psi(t)$.

It remains to see that the formulas valid in $\mathcal{K}_\Gamma$ are exactly the intuitionistically provable ones. This is the role of the truth lemma.

Lemma 6 (Truth Lemma)**.** For any formulas φ and ψ,

$$\overline{\varphi} \Vdash_{\nu_K} \psi \quad \Longleftrightarrow \quad \Gamma, \varphi \vdash \psi$$

Proof. The proof is by induction on the complexity of ψ. The basic cases hold by definition of $\mathcal{K}_\Gamma$. Due to space restrictions, we only provide here the most interesting inductive step, namely that for the existential quantifier. The other cases can be proved straightforwardly and constructively: in each case, the left-to-right direction of the lemma amounts to the introduction rule for the logical constant at stake, and the right-to-left direction to the elimination rule.

Existential quantifier $\Rightarrow$ Suppose $\overline{\varphi} \Vdash_{\nu_K} \exists x \psi(x)$. This means that there is a family $\{\overline{\varphi_i} \mid, i \in \mathcal{I}\}$ and elements $t_i^{\varphi_i} \in |M_{\overline{\varphi_i}}|$ such that $\overline{\varphi_i} \Vdash_{\nu_K[x \mapsto t_i^{\varphi_i}]} \psi(x)$ for all $i \in \mathcal{I}$. By remark 5 this is equivalent to $\overline{\varphi_i} \Vdash_{\nu_K} \psi(t_i)$, which by induction hypothesis amounts to $\Gamma, \varphi_i \vdash \psi(t_i)$.

Then, the rule of introduction of the existential quantifier guarantees that for any $i \in \mathfrak{I}$ we have $\Gamma, \varphi_i \vdash \exists x \psi(x)$. Finally, since $\overline{\varphi} \lhd \{\overline{\varphi_i} \,|, i \in \mathfrak{I}\}$, by the meaning of the covering relation we have $\Gamma, \varphi \vdash \exists x \psi(x)$.

$\Leftarrow$ Suppose $\Gamma, \varphi \vdash \exists x \psi(x)$. It is now up to us to produce a covering family and witnesses for the existential on each piece of the covering.

Our candidate is the family $\{\overline{\varphi \wedge \psi(y)} \,|\, y$ a variable$\}$, with the element $y^{\varphi \wedge \psi(y)}$ as a local witness at $\varphi \wedge \psi(y)$.

First, we have $\Gamma, \varphi \wedge \psi(y) \vdash \psi(y)$, whence by induction hypothesis $\overline{\varphi \wedge \psi(y)} \Vdash_{\nu_K} \psi(y)$, which by remark 5 is the same as $\overline{\varphi \wedge \psi(y)} \Vdash_{\nu_K[x \mapsto y^{\varphi \wedge \psi(y)}]} \psi(x)$.

This shows that the elements $y^{\varphi \wedge \psi(y)}$ act as local witnesses for the existential. It remains to show that the given family indeed covers $\overline{\varphi}$.

In order to see this, suppose a formula ξ is derivable from $\Gamma, \varphi \wedge \psi(y)$ for any variable y. Let y_0 be a variable that occurs neither in φ nor in ξ: in particular, we have $\Gamma, \varphi \wedge \psi(y_0) \vdash \xi$, that is, $\Gamma, \varphi, \psi(y_0) \vdash \xi$. But since y_0 occurs neither in Γ, φ nor in ξ (recall that Γ is a set of sentences) and since we had $\Gamma, \varphi \vdash \exists x \psi(x)$ by assumption, the rule of elimination of the existential quantifier allows us to conclude $\Gamma, \varphi \vdash \xi$.

This shows that indeed $\overline{\varphi} \lhd \{\overline{\varphi \wedge \psi(y)} \,|\, y$ a variable$\}$ and thus that $\overline{\varphi} \Vdash_{\nu_K} \exists x \psi(x)$, completing the proof of the inductive step.

□

The truth lemma speedily leads to a constructive proof of the strong completeness of intuitionistic logic for presheaf semantics.

Theorem 7 (Completeness theorem)**.** If $\Gamma \models_{ps} \varphi$, then $\Gamma \vdash \varphi$.

Proof. Suppose $\Gamma \models_{ps} \varphi$. The truth lemma shows that in the presheaf model $\mathcal{K}_\Gamma$ we have $\overline{\top} \Vdash_{\nu_K} \psi$ iff $\Gamma, \top \vdash \psi$, i.e. iff $\Gamma \vdash \psi$. In particular, $\overline{\top} \Vdash_{\nu_K} \gamma$ for any $\gamma \in \Gamma$ and thus also $\overline{\top} \Vdash_{\nu_K} \varphi$, whence again by the truth lemma we have $\Gamma \vdash \psi$.

References

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