# Bisimulation in Inquisitive Modal Logic

Ivano Ciardelli
Institute for Logic, Language and Computation
University of Amsterdam

Martin Otto
Department of Mathematics, Logic Group
Technische Universität Darmstadt

***Abstract*—Inquisitive modal logic, INQML, is a generalisation of standard Kripke-style modal logic. In its epistemic incarnation, it extends standard epistemic logic to a higher stratum of cognitive phenomena, capturing not only the concepts of *factual truth* (in possible worlds) and *knowledge* (in sets of possible worlds, a.k.a. information states), but also the concepts of *questions* and cognitive *issues* (in families of information states). Technically, INQML fits within the family of logics based on team semantics. From a model-theoretic perspective, it takes us a step in the direction of monadic second-order logic, as modal operators involve quantification over sets of worlds.**

**We introduce and investigate the natural notion of bisimulation equivalence in this model-theoretic scenario. We compare the expressiveness of INQML and first-order logic in the context of relational structures with two sorts, for worlds and information states, and characterise both basic inquisitive modal logic and its multi-agent epistemic (S5-like) variant as the bisimulation invariant fragments of first-order logic over corresponding classes of structures. These characterisations crucially involve non-classical methods in studying bisimulations and first-order expressiveness over non-elementary classes, irrespective of whether we aim for characterisations in the sense of classical or of finite model theory.**

## I. INTRODUCTION

The recently developed framework of *inquisitive logic* ([7], [2], [6], [4]) can be seen as a generalisation of classical logic which encompasses not only statements, but also questions.

One reason why this generalisation is interesting is that it provides a novel perspective on the logical notion of *dependency*, which plays an important role in applications (e.g., in database theory) and which has recently received much attention in the field of *dependence logic* ([19], among others). Indeed, dependency is nothing but a facet of the fundamental logical relation of entailment, once this is extended so as to apply not only to statement, but also to questions [4]. This connection explains the deep similarities existing between systems of inquisitive logic and systems of dependence logic (see [21], [3]). A different role for questions in a logical system comes from the setting of modal logic: once the notion of a modal operator is suitably generalised, questions can be embedded under modal operators to produce new statements that have no "standard" counterpart. This approach was first developed in [8] in the setting of epistemic logic. In the resulting *inquisitive epistemic logic* (IEL), agents are described as equipped not just with some information, but also with some issues that they would like to resolve. Modal formulae in IEL can express not only that an agent knows that $p$ ($\Box p$) but also that she knows *whether* $p$ ($\Box ?p$) or that she *entertains the question whether* $p$ ($\boxplus ?p$)—a statement that cannot be expressed without the use of embedded questions. As shown in [8], the most important notions of epistemic logic generalise smoothly to questions: besides common knowledge we now have *common issues*, the issues publicly entertained by the group; and besides publicly announcing a statement, agents can now also publicly ask a question, which typically results in new public issues being raised. Thus, inquisitive epistemic logic may be seen as one step in extending modal logic from a framework to reason about information and information change, to a richer framework which also represents a higher stratum of cognitive phenomena, such as issues and their raising in a communication scenario.

Of course, like standard modal logic, inquisitive modal logic INQML provides a general framework that admits various concrete interpretations, each suggesting corresponding constraints on the class of models. In addition to the epistemic one, [2] suggests an interpretation of inquisitive modal logic as a logic of action. In this interpretation, a formula $\Box ?p$ expresses that whether a certain fact $p$ will come about is determined independently of the agent's choices, while $\boxplus ?p$ expresses that whether $p$ will come about is fully determined by her choices.

From the perspective of mathematical logic, inquisitive modal logic is a natural generalisation of standard modal logic. There, the accessibility relation of a Kripke model associates with each possible world $w \in W$ a set $\sigma(w) \subseteq W$ of possible worlds, namely, the worlds accessible from $w$; any formula $\varphi$ of modal logic is semantically associated with a set $|\varphi|_{\mathbb{M}} \subseteq W$ of worlds, namely, the set of worlds where it is true; modalities then express relationships between these sets: for instance, $\Box\varphi$ expresses the fact that $\sigma(w) \subseteq |\varphi|_{\mathbb{M}}$. In the inquisitive setting, the situation is analogous, but both the entity $\Sigma(w)$ attached to a possible world and the semantic extension $[\varphi]_{\mathbb{M}}$ of a formula are sets *of sets* of worlds, rather than simple sets of worlds. Inquisitive modalities still express relationships between these two objects: $\Box\varphi$ expresses the fact that $\bigcup \Sigma(w) \in [\varphi]_{\mathbb{M}}$, while $\boxplus\varphi$ expresses the fact that $\Sigma(w) \subseteq [\varphi]_{\mathbb{M}}$.

In this manner, inquisitive logic leads to a new framework for modal logic that can be viewed as a generalisation of the standard framework. Clearly, this raises the question of whether and how the classical notions and results of modal logic carry over to this more general setting. In this paper we

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address this question for the fundamental notion of *bisimulation* and for two classical results revolving around this notion, namely, the Ehrenfeucht-Fraïssé theorem for modal logic, and characterisation theorems in the style of van Benthem's theorem [20]. For these hallmarks of modal model theory and for a wider model-theoretic discussion of the semantics of modal logics we refer to standard textbooks like [1], and in particular to [11]. A central topic of this paper is the rôle of *bisimulation invariance* as a unifying semantic feature that distinguishes modal logics—standard modal logic as well as many of its extensions developed for specific applications—from classical predicate logics. As in many other areas, ranging from temporal logics to process logics and from knowledge representation in AI to database applications, so also in the inquisitive setting we find that the appropriate notion of bisimulation invariance allows for precise model-theoretic characterisations of the expressive power of modal logic in relation to corresponding versions of predicate logic.

Our first result is that the right notion of inquisitive bisimulation equivalence $\sim$, with finitary approximation levels $\sim^n$, supports a counterpart for INQML of the classical Ehrenfeucht–Fraïssé correspondence. This result is non-trivial in the inquisitive setting, because of some subtle issues stemming from the interleaving of first- and second-order features in inquisitive modal logic.

**Theorem 0** (inquisitive Ehrenfeucht–Fraïssé theorem)**.** *Over finite vocabularies, the finite levels $\sim^n$ of inquisitive bisimulation equivalence correspond to the levels of* INQML*-equivalence up to modal nesting depth $n$.*

In order to compare INQML with classical first-order logic, we define a class of two-sorted relational structures, and show how such structures can encode models for INQML. With respect to such relational structures we find not only a "standard translation" of INQML into two-sorted first-order logic, but also several van Benthem style characterisations of INQML as the bisimulation-invariant fragment of (two-sorted) first-order logic. These results are technically interesting, and they are not available on the basis of classical techniques, because the relevant classes of two-sorted models are non-elementary (in fact, first-order logic is not compact over these classes, as we show). Our techniques yield characterisation theorems both in the setting of arbitrary inquisitive models, and in restriction to just finite ones.

**Theorem 1.** *Inquisitive modal logic can be characterised as the $\sim$-invariant fragment of first-order logic* FO *over natural classes of (finite or arbitrary) relational inquisitive models.*

We go on to extend these results from the basic inquisitive modal setting to the setting of inquisitive epistemic logic—the inquisitive counterpart of multi-agent S5. This setting is technically more challenging due to the additional S5-type constraints imposed on models.

**Theorem 2.** *Inquisitive epistemic logic (in a multi-agent setting) can be characterised as the $\sim$-invariant fragment of* FO *over natural classes of (finite or arbitrary) relational inquisitive epistemic models.*

Beside the conceptual development and the core results themselves, we think that also the methodological aspects of the present investigations have some intrinsic value. Just as inquisitive logic models cognitive phenomena at a level strictly above that of standard modal logic, so the model-theoretic analysis moves up from the level of ordinary first-order to a level strictly between first- and second-order logic. This level is realised by first-order logic in a two-sorted framework that incorporates second-order objects in the second sort in a non-trivially controlled fashion. We thus have to investigate both bisimulation equivalence and first-order expressiveness in the spectrum between first-order and monadic second-order logic, and work with first-order logic over characteristically non-elementary classes. In this scenario, Ehrenfeucht–Fraïssé games combine first-order locality techniques with limited and local recourse to monadic second-order features—features which, unconstrained, could be expected to defeat any attempt at a locality-based analysis. This leads us to substantially generalise a number of notions and techniques developed in the model-theoretic analysis of modal logic ([14], [9], among others), such as characteristic formulae, locally bisimilar unfoldings, the upgrading of back&forth equivalences, and bisimulation-preserving model transformations in the context of non-elementary frame classes.

*Structure of the paper:* Section II provides an essential introduction to inquisitive modal logic. Section III introduces the inquisitive notion of bisimilarity and its finite approximations, leading up to our first main result, the inquisitive Ehrenfeucht–Fraïssé theorem. Section IV introduces an encoding of inquisitive modal models as two-sorted relational structures; in the context of such structures we can regard INQML as a fragment of first-order logic, and we can study bisimulation invariance as a semantic constraint in Section V. The main technical results, viz. our characterisation theorems Theorems 1 and 2, are obtained in Sections VI and VII, respectively. For some technical arguments and details these sections are complemented by extra material in an appendix, Section VIII.

## II. INQUISITIVE MODAL LOGIC

In this section, we provide an essential introduction to inquisitive modal logic, INQML [2]. Since this section only sets the stage for our contribution, proofs are omitted. Obviously, our presentation here must be extremely succinct; for a more detailed introduction to these ideas, see §7 of [2].

### A. Foundations of inquisitive semantics

In inquisitive logics, statements and questions are interpreted in a uniform way. This is achieved by means of a novel perspective on semantics akin to the passage from standard semantics to *team semantics* in classical propositional or first-order logics [19], [22]. Usually, the semantics for a logic specifies truth-conditions for the formulae of the logic. In modal logics these truth-conditions are based on possible worlds in a Kripke model. However, this approach is limited in an important way: while suitable for statements, it seems inadequate for questions. To overcome this limitation, inquisitive logic interprets formulae not relative to states of affairs (worlds), but relative to states of information. Rather than specifying when a sentence is *true* at a world $w$, inquisitive semantics specifies when a sentence is *supported* by an information state $s$: for a statement $\alpha$ this means that the information available in $s$ implies that $\alpha$ is true; for a question $\mu$, it means that the information available in $s$ suffices to settle $\mu$.

As customary in modal logic since the work of Hintikka [13], an *information state* is extensionally modelled as a set $s \subseteq W$ of possible worlds, viz., the set of those worlds that are compatible with the available information.

**Definition 3** (information states)**.**
*An* information state *over a set of worlds $W$ is a subset $s \subseteq W$.*

If $t \subseteq s$, where $t$ and $s$ are information states, this means that $t$ holds at least as much information as $s$: with a view to information content, we say that $t$ is an *extension* of $s$. If $t$ is an extension of $s$, then everything that is supported at $s$ will also be supported at $t$. This property is a key feature of inquisitive semantics, and it leads naturally to the following notion of an *inquisitive state* (for a systematic discussion of the role of these semantic structures, see [5], [17], [8]).

**Definition 4** (inquisitive states)**.**
*An* inquisitive state *over a set of possible worlds $W$ is a non-empty set of information states $\Pi \subseteq \wp(W)$ satisfying*

- *$s \in \Pi$ and $t \subseteq s$ implies $t \in \Pi$ (downward closure).*

In the next section we will see how these fundamental notions are applied in the specific setting of inquisitive modal logic.

### *B. Inquisitive modal models*

A Kripke frame can be thought of as a set $W$ of worlds together with a map $\sigma$ that equips each world with a set of worlds $\sigma(w)$—the set of worlds that are accessible from $w$—which we may think of as an information state.

Similarly, an inquisitive modal frame consists of a set $W$ of worlds together with an *inquisitive assignment*, i.e., a map $\Sigma$ that assigns to each world an inquisitive state (a downward closed set of information states). An inquisitive modal model is an inquisitive frame equipped with a valuation function $V$ that associates to each atomic formula $p \in \mathcal{P}$ the set of worlds $V(p)$ at which the formula is true.

**Definition 5** (inquisitive modal models)**.**
*An inquisitive modal frame is a pair $\mathbb{F} = \langle W, \Sigma \rangle$, where $\Sigma\colon W \to \wp\wp(W)$ associates to each world $w \in W$ an inquisitive state $\Sigma(w)$. An inquisitive modal model is a pair $\mathbb{M} = \langle \mathbb{F}, V \rangle$ where $\mathbb{F}$ is an inquisitive modal frame, and $V\colon \mathcal{P} \to \wp(W)$ is a propositional valuation function. A world- (or state-)pointed inquisitive modal model is a pair consisting of a model $\mathbb{M}$ and a distinguished world (or information state) in this model.*

With an inquisitive modal model $\mathbb{M}$ we can always associate a standard Kripke model $\mathfrak{K}(\mathbb{M})$, over the same set of worlds and with the modal accessibility map $\sigma$ induced by the inquisitive state map $\Sigma$ according to

$$\begin{array}{rccl} \sigma: & W & \longrightarrow & \wp(W) \\ & w & \longmapsto & \sigma(w) := \bigcup \Sigma(w). \end{array}$$

A natural interpretation for inquisitive modal models is the epistemic one, developed in [8]. In this interpretation, the map $\Sigma$ is taken to describe not only an agent's *knowledge*, as in standard epistemic logic, but also an agent's *issues*. The agent's knowledge state at $w$, $\sigma(w) = \bigcup \Sigma(w)$, consists of all the worlds that are compatible with what the agent knows. The agent's inquisitive state at $w$, $\Sigma(w)$, consists of all those information states where the agent's issues are settled. This interpretation is particularly interesting in the multi-modal setting, where a model comes with multiple state maps $\Sigma_a$, one for each agent $a$ in a set $\mathcal{A}$. Moreover, this specific interpretation suggests some constraints on the maps $\Sigma_a$, analogous to the usual S5 constraints on Kripke models.

**Definition 6** (inquisitive epistemic models)**.**
*An inquisitive epistemic frame for a set $\mathcal{A}$ of agents is a pair $\mathbb{F} = \langle W, (\Sigma_a)_{a \in \mathcal{A}} \rangle$, where each map $\Sigma_a : W \to \wp\wp(W)$ associates with each world $w$ an inquisitive state $\Sigma_a(w)$ in accordance with the following constraints ($\sigma_a(w) := \bigcup \Sigma_a(w)$):*

- *$w \in \sigma_a(w)$ (factivity);*
- *$v \in \sigma_a(w) \Rightarrow \Sigma_a(v) = \Sigma_a(w)$ (full introspection).*

*An inquisitive epistemic model consists of an inquisitive epistemic frame together with a propositional assignment $V : W \to \wp(\mathcal{P})$.*

It is easy to verify that the Kripke frame associated with an inquisitive epistemic frame is an $S5$ frame, i.e., the accessibility maps $\sigma_a$ correspond to accessibility relations $R_a := \{(v, w) : v \in \sigma_a(w)\}$ that are equivalence relations on $W$.

### *C. Inquisitive modal logic*

The syntax of inquisitive modal logic InqML is given by:

$$\varphi ::= p \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \to \varphi) \mid (\varphi \mathbin{\vee\!\!\!\vee} \varphi) \mid \Box\varphi \mid \boxplus\varphi$$

We treat negation and disjunction as defined connectives (syntactic shorthands) according to $\neg\varphi := \varphi \to \bot$, and $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$. In this sense, the above syntax embeds standard propositional formulae in terms of atoms and connectives $\wedge$ and $\to$ together with the defined $\neg$ and $\vee$. As we will see, the semantics for such formulae will be essentially the same as in standard propositional logic. In addition to standard connectives, our language contains a new connective, $\mathbin{\vee\!\!\!\vee}$, called *inquisitive disjunction*. We may read formulae built up by means of this connective as propositional questions. E.g., we read the formula $p \mathbin{\vee\!\!\!\vee} \neg p$ as the question *whether or not* $p$, and we abbreviate this formula as $?p$. Finally, our language contains two modalities, which are allowed to embed both statements and questions. As we shall see, both these modalities coincide with a standard Kripke box when applied to statements, but crucially differ when applied to questions.

Under the epistemic interpretation of the logic, $\Box ?p$ expresses the fact that the agent knows whether $p$, while $\boxplus ?p$ expresses the fact that the agent entertains the issue whether $p$.

While models for INQML are formally a class of neighbourhood models, the semantics of INQML is very different from neighbourhood semantics for modal logic.[1] As mentioned above, the fully compositional semantics for INQML is based on the notion of support by an information state, rather than on truth at a possible world.

**Definition 7** (support semantics for INQML)**.**
*Let $\mathbb{M} = \langle W, \Sigma, V \rangle$ be an inquisitive modal model, $s \subseteq W$:*

- $\mathbb{M}, s \models p \iff s \subseteq V(p)$
- $\mathbb{M}, s \models \bot \iff s = \emptyset$
- $\mathbb{M}, s \models \varphi \wedge \psi \iff \mathbb{M}, s \models \varphi$ *and* $\mathbb{M}, s \models \psi$
- $\mathbb{M}, s \models \varphi \to \psi \iff \forall t \subseteq s : \mathbb{M}, t \models \varphi \Rightarrow \mathbb{M}, t \models \psi$
- $\mathbb{M}, s \models \varphi \mathbin{\mathbb{V}} \psi \iff \mathbb{M}, s \models \varphi$ *or* $\mathbb{M}, s \models \psi$
- $\mathbb{M}, s \models \Box\varphi \iff \forall w \in s : \mathbb{M}, \sigma(w) \models \varphi$
- $\mathbb{M}, s \models \boxplus\varphi \iff \forall w \in s\, \forall t \in \Sigma(w) : \mathbb{M}, t \models \varphi$

As an illustration, consider the support conditions for the formula $?p$, which abbreviates $p \mathbin{\mathbb{V}} \neg p$: this formula is supported by a state $s$ in case $p$ is true at all worlds in $s$ (i.e., if the information available in $s$ implies that $p$ is true) or in case $p$ is false at all worlds in $s$ (i.e., if the information available in $s$ implies that $p$ is false). Thus, $?p$ is supported precisely by those information states that settle whether or not $p$ is true.

The following two properties hold generally in INQML:

- Persistency: if $\mathbb{M}, s \models \varphi$ and $t \subseteq s$, then $\mathbb{M}, t \models \varphi$;
- Semantic ex-falso: $\mathbb{M}, \emptyset \models \varphi$ for all $\varphi \in$ INQML.

The first principle says that the support relation is preserved as information increases, i.e., as we move from a state to an extension of it. The second principle says that the empty set of worlds—the inconsistent information state—vacuously supports everything. Together, these principles imply that the support set $[\varphi]_{\mathbb{M}} := \{s \subseteq W : \mathbb{M}, s \models \varphi\}$ of a formula is downward closed and non-empty, i.e., it is an inquisitive state.

Even though the primary notion of our semantics is support with respect to an information state, truth at a world can be derived as a defined notion: a formula is true at $w$ if it is supported by the corresponding singleton state.

**Definition 8** (truth)**.** *We say that $\varphi$ is* true *at a world $w$ in a model $\mathbb{M}$, denoted $\mathbb{M}, w \models \varphi$, in case $\mathbb{M}, \{w\} \models \varphi$.*

Definition 8 together with our support conditions delivers the following truth conditions for our logic.

**Proposition 9** (derived truth conditions)**.**

- $\mathbb{M}, w \models p \iff p \in V(w)$
- $\mathbb{M}, w \not\models \bot$
- $\mathbb{M}, w \models \varphi \wedge \psi \iff \mathbb{M}, w \models \varphi$ *and* $\mathbb{M}, w \models \psi$
- $\mathbb{M}, w \models \varphi \to \psi \iff \mathbb{M}, w \not\models \varphi$ *or* $\mathbb{M}, w \models \psi$
- $\mathbb{M}, w \models \varphi \mathbin{\mathbb{V}} \psi \iff \mathbb{M}, w \models \varphi$ *or* $\mathbb{M}, w \models \psi$
- $\mathbb{M}, w \models \Box\varphi \iff \mathbb{M}, \sigma(w) \models \varphi$
- $\mathbb{M}, w \models \boxplus\varphi \iff \forall t \in \Sigma(w) : \mathbb{M}, t \models \varphi$

Notice that this does not provide a recursive definition of truth, since the truth conditions for modal formulae $\Box\varphi$ and $\boxplus\varphi$ crucially depend on the support conditions for the formula $\varphi$, and not just on its truth conditions.

For many formulae, support at a state boils down to truth at each world. We refer to these formulae as *truth-conditional*, as their semantics is fully determined by their truth-conditions.[2]

**Definition 10** (truth-conditional formulae)**.**
*We call $\varphi$* truth-conditional *if for each model $\mathbb{M}$ and state $s$: $\mathbb{M}, s \models \varphi \iff \mathbb{M}, w \models \varphi$ for all $w \in s$.*

Following [2], we think of truth-conditional formulae as statements, and of formulae that are not truth-conditional as questions. The next proposition identifies a large class of truth-conditional formulae.

**Proposition 11.** *Atomic formulae, $\bot$, and all formulae of the form $\Box\varphi$ and $\boxplus\varphi$ are truth-conditional. The class of truth-conditional formulae is closed under all connectives except for $\mathbb{V}$.*

Using this fact, it is easy to see that all standard modal formulae, i.e., formulae which do not contain $\mathbb{V}$ or $\boxplus$, receive exactly the same truth conditions as in standard modal logic.

**Proposition 12.** *If $\varphi$ is a standard modal formula, then $\mathbb{M}, w \models \varphi \iff \mathfrak{K}(\mathbb{M}), w \models \varphi$ in standard Kripke semantics.*

In fact, as long as questions are not around, the modality $\boxplus$ also coincides with $\Box$, and with the standard box modality.

**Proposition 13.** *If $\varphi$ is truth-conditional, $M, w \models \Box\varphi \iff M, w \models \boxplus\varphi \iff M, v \models \varphi$ for all $v \in \sigma(w)$*

Thus, the two modalities coincide on statements. However, they come apart when they are applied to questions. For an illustration, consider the formulae $\Box ?p$ and $\boxplus ?p$ in the epistemic setting. $\Box ?p$ is true iff the information state of the agent, $\sigma(w)$, settles the question $?p$; thus, $\Box ?p$ expresses the fact that the agent knows whether $p$. By contrast, $\boxplus ?p$ is true iff any information state that settles the agent's issues, i.e., any state $t \in \Sigma(w)$, also settles $?p$; thus $\boxplus ?p$ expresses that settling the question whether $p$ is part of the agent's goals.

It is worth pointing out that the presence of questions is essential for the expressive power of the system, even with respect to statements, i.e., truth-conditional formulae. For instance, $\boxplus ?p$ is truth-conditional, but [2] (Prop. 7.1.18) shows that this formula is not equivalent to any $\mathbb{V}$-free formula. Thus, the possibility of embedding questions under modalities allows us to capture properties of world-pointed inquisitive modal models that are otherwise inexpressible.

[1] This very different perspective on the models also leads us to a notion of bisimulation which is different from the one that has been considered for neighbourhood models (see [12]).

[2] In the dependence logic literature (e.g., [19], [22]), truth-conditional formulae are referred to as *flat* formulae.

## III. Bisimulation in inquisitive modal logic

### A. Inquisitive bisimulation equivalence

An inquisitive modal model can be seen as a structure with two sorts of entities, worlds and information states, which interact with each other. On the one hand, an information state $s$ is completely determined by the worlds that it contains; on the other hand, a world $w$ is determined by the atoms it makes true and the information states which lie in $\Sigma(w)$. Taking a more behavioural perspective, we can look at an inquisitive modal model as a model where two kinds of transitions are possible: from an information state $s$, we can make a transition to a world $w \in s$, and from a world $w$, we can make a transition to an information state $s \in \Sigma(w)$. This suggests a natural notion of bisimilarity, together with its natural finite approximations of $n$-bisimilarity for $n \in \mathbb{N}$. As usual, these notions can equivalently be defined either in terms of back-and-forth systems or in terms of strategies in corresponding bisimulation games. We chose the latter for its more immediate and intuitive appeal to the underlying dynamics of a "probing" of behavioural equivalence.

The game is played by two players, **I** and **II**, who act as challenger and defender of a similarity claim involving a pair of worlds $w$ and $w'$ or information states $s$ and $s'$ over two models $\mathbb{M} = \langle W, \Sigma, V\rangle$ and $\mathbb{M}' = \langle W', \Sigma', V'\rangle$. We denote world-positions as $\langle w, w'\rangle$ and state-positions as $\langle s, s'\rangle$, where $w \in W, w' \in W'$ and $s \in \wp(W), s' \in \wp(W')$, respectively. The game proceeds in rounds that alternate between world-positions and state-positions. Playing from a world-position $\langle w, w'\rangle$, **I** chooses an information state in the inquisitive state associated to one of these worlds ($s \in \Sigma(w)$ or $s' \in \Sigma'(w')$) and **II** must respond by choosing an information state on the opposite side, which results in a state-position $\langle s, s'\rangle$. Playing from a state-position $\langle s, s'\rangle$, **I** chooses a world in one of these states ($w \in s$ or $w' \in s'$) and **II** must respond by choosing a world from the other state, which results in a world-position $\langle w, w'\rangle$. We take a round of the game to consist of four moves leading from a world-position to another.

In the bounded version of the game, the number of rounds is fixed in advance. In the unbounded version, the game is allowed to go on indefinitely. Either player loses when stuck for a move. The game ends with a loss for **II** in any world-position $\langle w, w'\rangle$ that shows a discrepancy at the atomic level, i.e., such that $w$ and $w'$ disagree on the truth of some $p \in \mathcal{P}$. All other plays, including infinite runs of the unbounded game, are won by **II**.

**Definition 14** (bisimulation equivalence)**.** *Two world-pointed models $\mathbb{M}, w$ and $\mathbb{M}', w'$ are* $n$-bisimilar*, $\mathbb{M}, w \sim^n \mathbb{M}', w'$, if* **II** *has a winning strategy in the $n$-round bisimulation game starting from $\langle w, w'\rangle$. $\mathbb{M}, w$ and $\mathbb{M}', w'$ are* bisimilar*, $\mathbb{M}, w \sim \mathbb{M}', w'$, if* **II** *has a winning strategy in the unbounded bisimulation game starting from $\langle w, w'\rangle$.*

*We say that two state-pointed models $\mathbb{M}, s$ and $\mathbb{M}', s'$ are ($n$-)bisimilar, denoted $\mathbb{M}, s \sim \mathbb{M}', s'$ or $\mathbb{M}, s \sim^n \mathbb{M}', s'$, if every world in $s$ is ($n$-)bisimilar to some world in $s'$ and vice versa.*

*Two models $\mathbb{M}$ and $\mathbb{M}'$ are* globally bisimilar*, denoted $\mathbb{M} \sim \mathbb{M}'$, if every world $w$ of $\mathbb{M}$ is bisimilar to some world $w'$ of $\mathbb{M}'$ and vice versa.*

These notions generalise naturally to the multi-modal setting with inquisitive assignments $(\Sigma_a)_{a \in \mathcal{A}}$ for a set $\mathcal{A}$ of agents, where at a world-position player **I** also gets the choice of which agent to probe.

### B. The Ehrenfeucht–Fraíssé theorem for InqML

The crucial rôle of these notions of equivalence for the model theory of inquisitive modal logic is brought out in a corresponding Ehrenfeucht–Fraíssé theorem.

Using the standard notion of the modal depth of a formula, we denote as $\mathrm{InqML}_n$ the class of InqML-formulae of depth up to $n$. It is easy to see that the semantics of any formula in $\mathrm{InqML}_n$ is preserved under $n$-bisimilarity; as a consequence, all of inquisitive modal logic is preserved under full bisimilarity. The corresponding analogue of the classical Ehrenfeucht–Fraíssé theorem shows that, in the case of finite sets $\mathcal{P}$ of basic propositions and finite sets $\mathcal{A}$ of agents, $n$-bisimilarity exactly coincides with logical indistinguishability in $\mathrm{InqML}_n$, which we denote as $\equiv^n_{\mathrm{InqML}}$:

$$\mathbb{M}, s \equiv^n_{\mathrm{InqML}} \mathbb{M}', s' \overset{\text{def}}{\iff} \begin{cases} \mathbb{M}, s \models \varphi \;\Leftrightarrow\; \mathbb{M}', s' \models \varphi \\ \qquad \text{for all } \varphi \in \mathrm{InqML}_n. \end{cases}$$

**Theorem 15** (Ehrenfeucht–Fraíssé theorem for InqML)**.** *Given a finite set of basic propositions $\mathcal{P}$ and (in the multi-modal case) a finite set $\mathcal{A}$ of agents, for any $n \in \mathbb{N}$ and inquisitive state-pointed modal models $\mathbb{M}, s$ and $\mathbb{M}', s'$:*

$$\mathbb{M}, s \sim^n \mathbb{M}', s' \iff \mathbb{M}, s \equiv^n_{\mathrm{InqML}} \mathbb{M}', s'$$

Notice that, by taking $s$ and $s'$ to be singleton states, we obtain the corresponding connection for world-pointed models as a special case: $\mathbb{M}, w \sim^n \mathbb{M}', w' \iff \mathbb{M}, w \equiv^n_{\mathrm{InqML}} \mathbb{M}', w'$. This analogue of the classical Ehrenfeucht–Fraïssé correspondence, which among many others also has natural variants for traditional modal logic and bisimulation, fits the richer format for inquisitive modal logic and bisimulation. As customary, the crucial implication, from right to left, follows from the existence of *characteristic formulae* that define $\sim^n$-classes of worlds, as well as related characteristic features of information states and inquisitive states over models—and it is here that the finiteness of $\mathcal{A}$ and $\mathcal{P}$ is crucial.

**Proposition 16** (characteristic formulae for $\sim^n$-classes)**.** *For any world-pointed model $\mathbb{M}, w$ over a finite set of basic propositions $\mathcal{P}$ and (in the multi-modal case) finite set $\mathcal{A}$ of agents, and for any $n \in \mathbb{N}$ there is a formula $\chi^n_{\mathbb{M},w} \in \mathrm{InqML}$ of modal depth $n$ s.t. $\mathbb{M}', w' \models \chi^n_{\mathbb{M},w} \iff \mathbb{M}', w' \sim^n \mathbb{M}, w$.*

*Proof.* We explicitly treat the mono-modal case, and generate formulae $\chi^n_{\mathbb{M},w}$ together with auxiliary formulae $\chi^n_{\mathbb{M},s}$ and $\chi^n_{\mathbb{M},\Pi}$ by simultaneous induction on $n$, for all worlds $w$, information states $s$ and inquisitive states $\Pi$ over $\mathbb{M}$. Given two inquisitive states $\Pi$ and $\Pi'$ in models $\mathbb{M}$ and $\mathbb{M}'$, we write $\mathbb{M}, \Pi \sim^n \mathbb{M}', \Pi'$ if every information state $s \in \Pi$ is

$n$-bisimilar to some information state $s' \in \Pi'$, and vice versa. Dropping annotations by $\mathbb{M}$ (which is fixed) we define:

$$\chi^0_w = \bigwedge\{p\colon w \in V(p)\} \wedge \bigwedge\{\neg p\colon w \notin V(p)\}$$
$$\chi^n_s = \bigvee\{\chi^n_w\colon w \in s\}$$
$$\chi^n_\Pi = \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}\{\chi^n_s\colon s \in \Pi\}$$
$$\chi^{n+1}_w = \chi^n_w \wedge \boxplus\chi^n_{\Sigma(w)} \wedge \bigwedge\{\neg\boxplus\chi^n_\Pi\colon \Pi \subseteq \Sigma(w),\, \Pi \not\sim^n \Sigma(w)\}$$

These formulae are of the required modal depth; the conjunctions and disjunctions in the definition are well-defined since, for a given $n$, there are only finitely many distinct formulae of the form $\chi^n_w$, and analogously for $\chi^n_s$ or $\chi^n_\Pi$. We can then prove by simultaneous induction on $n$ that these formulae satisfy the following properties:

1) $\mathbb{M}', w' \models \chi^n_{\mathbb{M},w} \iff \mathbb{M}', w' \sim^n \mathbb{M}, w$
2) $\mathbb{M}', s' \models \chi^n_{\mathbb{M},s} \iff \mathbb{M}', s' \sim^n \mathbb{M}, t$ for some $t \subseteq s$
3) $\mathbb{M}', s' \models \chi^n_{\mathbb{M},\Pi} \iff \mathbb{M}', s' \sim^n \mathbb{M}, s$ for some $s \in \Pi$

That is, the characteristic formula $\chi^n_{\mathbb{M},w}$ for a world-pointed model $\mathbb{M}, w$ precisely characterises the $\sim^n$-type of $\mathbb{M}, w$. As for the auxiliary formulae, $\chi^n_{\mathbb{M},s}$ characterises the $\sim^n$-type of the state-pointed model $\mathbb{M}, s$ up to compliance with downward closure; $\chi^n_{\mathbb{M},\Pi}$ precisely characterises the set of $\sim^n$-types of information states in the inquisitive state $\Pi$. The details of the inductive proof are given in the appendix, Section VIII-A. $\square$

**Definition 17.** *We say that a class $\mathcal{C}$ of world-pointed models is* defined *by a formula $\varphi \in$* INQML *if $\mathcal{C}$ is exactly the set of world-pointed models where $\varphi$ is true.*

**Corollary 18.** *A class $\mathcal{C}$ of world-pointed models for finite $\mathcal{P}$ and $\mathcal{A}$ is definable in* INQML *if and only if it is closed under $\sim^n$ for some $n \in \mathbb{N}$.*

For classes of state-pointed models, the situation is slightly more subtle. Due to persistency, a definable class $\mathcal{C}$ of state-pointed Kripke models must be downward closed in the sense that $\mathbb{M}, s \in \mathcal{C}$ implies $\mathbb{M}, t \in \mathcal{C}$ for all $t \subseteq s$.

**Corollary 19.** *A class $\mathcal{C}$ of state-pointed models for finite $\mathcal{P}$ and $\mathcal{A}$ is definable in* INQML *if and only if it is both downward closed and closed under $\sim^n$ for some $n \in \mathbb{N}$.*

## IV. Relational inquisitive models

### A. Relational inquisitive modal models

In this paper, we want to compare the expressive power of inquisitive modal logic with that of first-order logic. However, this is not straightforward. A standard Kripke model can be identified naturally with a relational structure with a binary accessibility relation $R$ (or $R_a$ for $a \in \mathcal{A}$) and a unary predicate $P_i$ for the interpretation of each atomic sentence $p_i \in \mathcal{P}$. By contrast, an inquisitive modal model also needs to encode the inquisitive state map $\Sigma : W \to \wp\wp(W)$. This map can be naturally identified with a binary relation $E \subseteq W \times \wp(W)$. In order to view this relation as part of a relational structure, however, we need to adopt a two-sorted perspective, and view the domains $W$ and $\wp(W)$ as domains of two distinct sorts. This leads us naturally to consider a new kind of structures.

**Definition 20** (relational models)**.** *A* relational inquisitive modal model *is a relational structure*

$$\mathfrak{M} = \langle W, S, E, \varepsilon, (P_i)_{p_i \in \mathcal{P}}\rangle$$

*where $W, S \neq \emptyset$ are sets, $E, \varepsilon \subseteq W \times S$, and $P_i \subseteq W$. With $s \in S$ we associate the set $\underline{s} := \{w \in W \mid w\varepsilon s\} \subseteq W$ and require the following conditions, which enforce resemblance with inquisitive modal models:*

- *Extensionality: if $\underline{s} = \underline{s}'$, then $s = s'$.*
- *Non-emptiness: for every $w$, $E[w] \neq \emptyset$.*
- *Downward closure: if $s \in E[w]$ and $t \subseteq \underline{s}$, there is an $s' \in S$ such that $\underline{s}' = t$ and $s' \in E[w]$.*

*Multi-modal variants are analogously defined, with relations $E_a \subseteq W \times S$ to encode the inquisitive assignments $\Sigma_a$ for the agents $a \in \mathcal{A}$. A relational inquisitive epistemic model is a multi-modal relational inquisitive model which satisfies the additional constraints of Definition 6, viz., factivity and full introspection.*

By extensionality, the second sort $S$ can be identified with a domain $\{\underline{s} \mid s \in S\} \subseteq \wp(W)$ of sets over the first sort. We will always make this identification and view a relational model as a structure $\mathfrak{M} = \langle W, S, E, \in, (P_i)\rangle$ where $S \subseteq \wp(W)$ and $\in$ is the actual membership relation. In the following we shall therefore also specify relational inquisitive models by just $\mathfrak{M} = \langle W, S, E, (P_i)\rangle$, when the fact that $S \subseteq \wp(W)$ and the natural interpretation of $\varepsilon$ are understood.

Notice that a relational model $\mathfrak{M}$ induces a corresponding Kripke model $\mathfrak{K}(\mathfrak{M})$ on $W$. We simply let $wRw'$ if for some $s \in S$ we have $wEs$ and $w'\varepsilon s$, and we let $R[w] := \{w' \mid wRw'\}$.

### B. Natural classes of relational models

In addition to extensionality and downward closure, we might impose other constraints on a relational model $\mathfrak{M}$: in particular, we may require $S$ to be the full powerset of $W$, or to resemble the powerset from the perspective of each world $w$.

**Definition 21** (classes of relational models)**.**
*A relational model $\mathfrak{M} = \langle W, S, E, (P_i)\rangle$ is called:*

- full *if $S = \wp(W)$;*
- locally full *if $S \supseteq \wp(R[w])$ for all $w \in W$.*

These conditions suggest different ways of encoding a concrete inquisitive modal model $M = \langle W, \Sigma, V\rangle$ as a relational model.

**Definition 22** (relational encodings)**.**
*Let $\mathbb{M} = \langle W, \Sigma, V\rangle$ be an inquisitive modal model. We define three relational encodings $\mathfrak{M}^{[\cdots]}(\mathbb{M})$ of $\mathbb{M}$, each based on $W$, and with $wEs \Leftrightarrow s \in \Sigma(w)$, $w\ \varepsilon\ s \Leftrightarrow w \in s$ and $P_i = V(p_i)$. The encodings differ in the second sort domain $S$:*

- *for $\mathfrak{M}^{\mathsf{rel}}(\mathbb{M})$: $S = \mathrm{image}(\Sigma)$;*
- *for $\mathfrak{M}^{\mathsf{lf}}(\mathbb{M})$: $S = \{s \subseteq \sigma(w)\colon w \in W\}$;*
- *for $\mathfrak{M}^{\mathsf{full}}(\mathbb{M})$: $S = \wp(W)$.*

*These definitions generalise naturally to the multi-modal case.*

Clearly, $\mathfrak{M}^{\mathsf{rel}}(\mathbb{M})$ is the minimal relational counterpart of $\mathbb{M}$, $\mathfrak{M}^{\mathsf{lf}}(\mathbb{M})$ its minimal counterpart that is locally full, and $\mathfrak{M}^{\mathsf{full}}(M)$ its unique counterpart that is full.

### C. Relational models and first-order logic

A relational inquisitive model supports a two-sorted first-order language having two relation symbols $\mathtt{E}$ and $\varepsilon$, and a number of predicate symbols $\mathtt{P}_i$ for $i \in I$. It is easy to translate formulae $\varphi \in \text{INQML}$ to FO-formulae $\varphi^*(x)$ in a single free variable $x$ of the second sort in such a way that, if $\mathbb{M}$ is an inquisitive modal model and $\mathfrak{M}(\mathbb{M})$ is any of the above encodings, we have:

$$\mathbb{M}, s \models \varphi \iff \mathfrak{M}(\mathbb{M}) \models \varphi^*(s)$$

This translation can be seen as an analogue of the standard translation of modal logic to first-order logic. The framework of relational inquisitive models thus allows us to view INQML as a syntactic fragment of FO, INQML $\subseteq$ FO, just as standard modal logic ML over Kripke structures may be regarded as a fragment ML $\subseteq$ FO.

Importantly, however, the class of relational inquisitive modal models is not first-order definable in this framework, since the downward closure condition involves a second-order quantification. In other words, we are dealing with first-order logic FO over non-elementary classes of intended models.

## V. Bisimulation invariance as a semantic constraint

Regarding INQML as a fragment of first-order logic (over relational models, in any one of the above classes), we may think of downward closure and $\sim$-invariance as characteristic semantic features of this fragment. The core question for much of the rest of this paper is to which extent INQML may express (i) all properties of worlds that are FO-expressible; and (ii) all properties of information states that are FO-expressible, $\sim$-invariant, and downward closed. In other words, over what classes $\mathcal{C}$ of models, if any, can INQML be characterised as the bisimulation invariant fragment of first-order logic? In short, for what classes $\mathcal{C}$ of relational models do we have

$$\text{INQML} \equiv \mathsf{FO}/\!\sim \qquad (\dagger)$$

just as ML $\equiv$ FO/$\sim$ by van Benthem's theorem?

### A. Bisimulation invariance and compactness

The inquisitive Ehrenfeucht–Fraïssé theorem, Theorem 15, implies $\sim$-invariance for all of INQML. By Corollaries 18 and 19 it further implies *expressive completeness* of $\text{INQML}_n$ for any $\sim^n$-invariant property of world-pointed models, and any downward-closed $\sim^n$-invariant property of state-pointed models. In order to prove ($\dagger$), say for world properties, and possibly in restriction to any particular class $\mathcal{C}$ of relational inquisitive models, it is thus necessary and sufficient to show that, for any $\varphi(x) \in \mathsf{FO}$, $\sim$-invariance of $\varphi(x)$ over $\mathcal{C}$ implies $\sim^n$-invariance of $\varphi(x)$ over $\mathcal{C}$ for some finite $n$. This may be viewed as a *compactness principle* for $\sim$-invariance of first-order properties, which is non-trivial in the non-elementary setting of relational inquisitive models.

**Observation 23.** *Given any class $\mathcal{C}$ of relational inquisitive models, the following are equivalent:*
*(i)* INQML $\equiv$ FO/$\sim$ *for world properties over $\mathcal{C}$;*
*(ii) for* FO*-properties of world-pointed models, $\sim$-invariance over $\mathcal{C}$ implies $\sim^n$-invariance over $\mathcal{C}$ for some $n$.*
*Similarly, the following are equivalent:*
*(i)* INQML $\equiv$ FO/$\sim$ *for downward-closed state properties over $\mathcal{C}$;*
*(ii) for downward closed* FO*-properties of state-pointed models, $\sim$-invariance over $\mathcal{C}$ implies $\sim^n$-invariance over $\mathcal{C}$ for some $n$.*

In the following sections, we explicitly restrict attention to properties of worlds, and mostly leave implicit the corresponding results for downward-closed state properties.

Interestingly, first-order logic does not satisfy compactness in restriction to the (non-elementary) class of relational inquisitive models (see Example 39 in the appendix). More importantly, over the class of *full* relational models, violations of compactness can even be exhibited for $\sim$-invariant formulae.

**Observation 24.** *Over full relational inquisitive models, the absence of infinite $R$-paths from the designated world $w$ (i.e., well-foundedness of the converse of $R$ at $w$) is a first-order definable and $\sim$-invariant property of worlds that is not preserved under $\sim^n$ for any $n$, hence not expressible in* INQML. *In particular, first-order logic violates compactness over full relational models (unlike* INQML*, which is known to satisfy compactness [2]).*

Due to Observation 23, this means that ($\dagger$) fails over the class of full relational models, which may not be too surprising, given that, on full relational models, FO has access to full-fledged monadic second-order quantification, whereas INQML can only quantify over subsets within the range of $\Sigma$.

### B. Upgrading of bisimulation levels

In light of Observation 23, to show that ($\dagger$) holds over a class $\mathcal{C}$ we need to show that a first-order formula $\varphi(x)$ whose semantics is invariant under $\sim$ over the class $\mathcal{C}$, is in fact invariant under one of the much coarser finite approximations $\sim^n$ over $\mathcal{C}$, for some value $n$ depending on $\varphi$. For this there is a general approach that has been successful in a number of similar investigations, starting from an elementary and constructive proof in [14] of van Benthem's classical characterisation of basic modal logic [20] and its finite model theory version due to Rosen [18] (for ramifications of this method, see also [15], [9] and [16]). This approach involves an *upgrading* of a sufficiently high finite level $\sim^n$ of bisimulation equivalence (or $\equiv^n_{\text{INQML}}$) to a finite target level $\equiv_q$ of elementary equivalence, where $q$ is the quantifier rank of $\varphi$. Concretely, this amounts to finding, for any world-pointed relational model $\mathfrak{M}, w$, a fully bisimilar pointed model $\mathfrak{M}^*, w^*$ with the property that,

$$\begin{tikzcd}
\mathfrak{M}, w \arrow[r, dash, "\sim^n"] \arrow[d, dash, "\sim"'] & \mathfrak{M}', w' \arrow[d, dash, "\sim"] \\
\mathfrak{M}^*, w^* \arrow[r, dash, "\equiv_q"] & \mathfrak{M}'^*, w'^*
\end{tikzcd}$$

Fig. 1. Generic upgrading pattern.

if $\mathfrak{M}, w \sim^n \mathfrak{M}', w'$, then $\mathfrak{M}^*, w^* \equiv_q \mathfrak{M}'^*, w'^*$. The diagram in Figure 1 shows how $\sim$-invariance of $\varphi$, together with its nature as a first-order formula of quantifier rank $q$, entails its $\sim^n$-invariance.

Any such upgrading involves an interesting tension between the very distinct levels of expressiveness of INQML-formulae and FO-formulae. While the latter can, for instance, distinguish worlds w.r.t. finite branching degrees of the accessibility relation $R$ or w.r.t. short cycles that $R$ may form in the vicinity of a world, no $\sim$-invariant logic can. The challenge is to overcome this discrepancy in bisimilar companion structures, using the malleability up to $\sim$ of (relational) inquisitive models (within the respective class!)—and, for instance, to boost all multiplicities and lengths of all cycles beyond what can be distinguished in $\mathsf{FO}_q$ (FO up to quantifier rank $q$).

In the next two sections, we shall exhibit $\sim$-preserving model transformations and constructions that allow us to achieve the required upgradings in different settings:

(1) for the class of all (finite) relational models, as well as the class of (finite) locally full models; this is described in Section VI, leading up to Theorem 1;

(2) for the class of (finite) relational epistemic models, as well as the class of (finite) locally full epistemic models; this is described in Section VII, leading up to Theorem 2.

For (1), we use a variation on an upgrading technique from [14], which is based on an inquisitive analogue of partial tree unfoldings; after this pre-processing, the models involved support locality arguments for first-order Ehrenfeucht–Fraïssé games (in effect we shall deviate slightly form the generic picture in Figure 1 by interleaving $\sim$-preserving pre-processing steps and $\equiv_q$-preserving steps). The classes of models in (2), on the other hand, do not allow for simple partial unfoldings and require a much more sophisticated analysis; in particular, some features of monadic second-order logic need to be taken more seriously, features that come into play through the presence of the second sort.

## VI. The $\sim$-invariant fragment of FO

In this section, we show that INQML can be characterised as the $\sim$-invariant fragment of first-order logic FO over natural classes of relational inquisitive models, in the following sense.

**Theorem 1**. *Let $\mathcal{C}$ be any of the following classes of relational models: the class of all models; of finite models; of locally full models; of finite locally full models. Over each of these classes,* INQML $\equiv$ FO$/\sim$, *i.e., a property of world-pointed models is definable in* INQML *over $\mathcal{C}$ if and only if it is both* FO*-definable over $\mathcal{C}$ and $\sim$-invariant over $\mathcal{C}$.*

The most useful tool from first-order model theory for our purposes is the *local nature* of first-order logic over relational structures, in terms of Gaifman distance, as given by Gaifman's theorem (cf. [10] and Theorem 34 below). For this section we just need the underlying notion of Gaifman distance and neighbourhoods: in a relational model, Gaifman distance is graph distance in the undirected bi-partite graph on the sets $W$ of worlds and $S$ of states with edges between any pair linked by $E$ or $\varepsilon$; the *$\ell$-neighbourhood* $N^\ell(w)$ of a world $w$ consists of all worlds or states at distance up to $\ell$ from $w$ in this sense. It is easy to see that if $\mathfrak{M}, w$ is a world-pointed relational model and $\ell \neq 0$ is even, the restriction of this model to $N^\ell(w)$, denoted $\mathfrak{M}{\restriction}N^\ell(w), w$, is also a world-pointed relational model.

### A. Partial unfolding and stratification

As discussed in the previous section, proving Theorem 1 boils down to showing the compactness property expressed in Observation 23 for the relevant classes of relational models. To show this property we make use of a process of *stratification*, comparable to tree-like unfoldings in standard modal logic.

**Definition 25.** We say that a relational inquisitive model $\mathfrak{M}$ is *stratified* if its two domains $W$ and $S$ consist of essentially disjoint[3] strata $(W_i)_{i\in\mathbb{N}}$ and $(S_i)_{i\in\mathbb{N}}$ s.t.
(i) $W = \dot{\bigcup} W_i$ and $S \setminus \{\emptyset\} = \dot{\bigcup}(S_i \setminus \{\emptyset\})$;[3]
(ii) $S_i \subseteq \wp(W_{i+1})$, and $E[w] \subseteq S_i$ for all $w \in W_i$.
For an even number $\ell \neq 0$, we say that $\langle W, S, E\rangle$ is *stratified to depth $\ell$ from $w$* if $\langle W, S, E\rangle \restriction N^\ell(w)$ is stratified.

It is not hard to see that any world-pointed relational inquisitive model is bisimilar to one that is stratified. Moreover, for any even number $\ell \neq 0$, a finite world-pointed relational model is bisimilar to one that is finite and stratified to depth $\ell$ from its distinguished world. The underlying process of partial unfolding preserves local fullness (on the other hand it violates fullness, as non-trivial full frames cannot be stratified).

**Observation 26.** *For relational models $\mathfrak{M}$ and $\mathfrak{M}'$ that are stratified to depth $\ell$ for some even $\ell \neq 0$, and for $n \geqslant \ell/2$:*

$$\begin{aligned} &\mathfrak{M}\restriction N^\ell(w), w \sim^n \mathfrak{M}'\restriction N^\ell(w'), w' \\ \Rightarrow\ &\mathfrak{M}\restriction N^\ell(w), w \sim \mathfrak{M}'\restriction N^\ell(w'), w'. \end{aligned}$$

This is because, due to stratification and cut-off, the $n$-round game fully exhausts all possibilities in the unbounded game.

### B. Proof of Theorem 1

Let $\mathcal{C}$ be any of the classes in Theorem 1 and let $\varphi(x) \in \mathsf{FO}_q$ be $\sim$-invariant over $\mathcal{C}$. We want to show that $\varphi$ is $\sim^n$-invariant over $\mathcal{C}$ for $n = 2^q$, where $q$ is the quantifier rank of $\varphi$. To this end, consider a world-pointed relational model $\mathfrak{M}, w$ in $\mathcal{C}$. Since $\varphi$ is $\sim$-invariant, we can assume w.l.o.g. that $\mathfrak{M}, w$ is stratified to depth $\ell = n$. We define two world-pointed models $\mathfrak{M}_0, w$ and $\mathfrak{M}_1, w$ as follows. Each of these models contains $q$ distinct isomorphic copies of $\mathfrak{M}$ as well as

[3]The $S_i$ will share the trivial information state $\emptyset$, by extensionality and the downward closure requirement. This is unproblematic for our purposes.

$$
\begin{tikzcd}
\mathfrak{M}, w \arrow[r, dash, "\sim^n"] \arrow[d, dash, "\sim"] & \mathfrak{M}', w' \arrow[d, dash, "\sim"] \\
\mathfrak{M}_1, w \arrow[r, dotted, dash, "\sim^n"] \arrow[d, dash, "\equiv_q"] & \mathfrak{M}'_1, w' \arrow[d, dash, "\equiv_q"] \\
\mathfrak{M}_0, w \arrow[r, dotted, dash, "\sim^n"] \arrow[d, dash, "\sim"] & \mathfrak{M}'_0, w' \arrow[d, dash, "\sim"] \\
\mathfrak{M}\restriction N^\ell(w), w \arrow[r, dash, "\sim"] & \mathfrak{M}'\restriction N^\ell(w'), w'
\end{tikzcd}
$$

Fig. 2. Upgrading pattern for Theorem 1.

of $\mathfrak{M}\restriction N^\ell(w)$.[4] In addition, $\mathfrak{M}_0$ contains a copy of $\mathfrak{M}\restriction N^\ell(w)$ with the distinguished world $w$, while $\mathfrak{M}_1$ contains a copy of $\mathfrak{M}$ with the distinguished world $w$:

$$
\begin{array}{lllll}
\mathfrak{M}_0, w := & q \otimes \mathfrak{M} & \oplus \; \mathfrak{M}\restriction N^\ell(w), w & \oplus & q \otimes \mathfrak{M}\restriction N^\ell(w) \\
\mathfrak{M}_1, w := & q \otimes \mathfrak{M} & \oplus \qquad \mathfrak{M}, w & \oplus & q \otimes \mathfrak{M}\restriction N^\ell(w)
\end{array}
$$

Using an Ehrenfeucht-Fraïssé game argument for $\mathsf{FO}$ it is not hard show (cf. Section VIII-C in the appendix) that

$$\mathfrak{M}_0, w \equiv_q \mathfrak{M}_1, w.$$

Given any two pointed models $\mathfrak{M}, w \sim^n \mathfrak{M}', w'$ in $\mathcal{C}$, we can then see that $\varphi$ is preserved between them, by chasing the diagram in Figure 2 along the path through the auxiliary models, which are all in $\mathcal{C}$.

## VII. The $\sim$-invariant epistemic fragment of $\mathsf{FO}$

In this section, we show that inquisitive epistemic logic, i.e., inquisitive multi-modal logic restricted to inquisitive epistemic models (cf. Definitions 6, 20 and 21), can be characterised as the $\sim$-invariant fragment of first-order logic $\mathsf{FO}$ over natural classes of relational inquisitive models, in the following sense.

**Theorem 2**. *Let $\mathcal{C}$ be any of the following classes of relational epistemic models: all or all finite; all or all finite locally full. Over each of these classes,* INQML $\equiv \mathsf{FO}/{\sim}$*, i.e., a property of world-pointed models is definable in* INQML *over $\mathcal{C}$ if and only if it is both $\mathsf{FO}$-definable over $\mathcal{C}$ and $\sim$-invariant over $\mathcal{C}$.*

To prove this result, we will again rely crucially on the locality of first-order logic. However, the stratification technique we used in the previous section to prepare our models for the game-theoretic proof will not do. This is because the stratification of a (relational) inquisitive epistemic model is not itself an inquisitive epistemic model. Instead, a much more sophisticated proof strategy is needed.

The claims over the classes of locally full relational epistemic models imply the claims for general relational epistemic models, since $\sim$-invariance over the larger classes implies $\sim$-invariance over the more restricted classes, which still represent every (finite) general relational model up to $\sim$. So we may concentrate on locally full encodings in the following.

[4] To comply with extensionality, we identify the empty information state across the otherwise disjoint union.

### A. Local view: the mono-modal case

Recall that an inquisitive epistemic model $\mathbb{M}$ induces a standard multi-modal $S5$ Kripke model $\mathfrak{K}(\mathbb{M})$, whose accessibility relations $R_a$ are equivalence relations. Now consider the $R_a$-equivalence class of a world $w$ in the model $\mathbb{M}$: we call this the *$a$-class* of $w$, and denote it by $[w]_a$. To the model $\mathbb{M}$ we can then associate its *local $a$-frames* which are the mono-modal inquisitive frames induced by $\mathbb{M}$ on $[w]_a$:

$$\mathbb{F}_a(w) := \langle [w]_a, \Sigma_a \restriction [w]_a \rangle.$$

Notice that, due to full introspection, the inquisitive state map of a local $a$-frame is constant, i.e., it assigns the same inquisitive state to each world $w' \in [w]_a$.

We are interested in expansions of these local inquisitive frames that keep track of the $\sim$-types or $\sim^n$-types that the worlds in $[w]_a$ have in $\mathbb{M}$. To this end we introduce expansions by new atomic propositions, which we label by corresponding equivalence classes $c \in \mathbb{M}/{\sim}$ or $c \in \mathbb{M}/{\sim^n}$. In the following we shall use letters $c$ for the new atomic propositions and think of them as disjoint colours for the $\sim$-types or $\sim^n$-types imported from $\mathbb{M}$. We also refer to their interpretations as *colour classes*: this interpretation is the natural one that assigns to $c$ the restriction of this subset $c \subseteq W$ to $[w]_a$:

$$
\begin{array}{ll}
\mathbb{M}_a^{\sim}(w) := \langle \mathbb{F}_a(w), V_a^{\sim} \rangle & \text{where } V_a^{\sim}(c) = c \cap [w]_a, \\
\mathbb{M}_a^{\sim^n}(w) := \langle \mathbb{F}_a(w), V_a^{\sim^n} \rangle & \text{where } V_a^{\sim^n}(c) = c \cap [w]_a,
\end{array}
$$

for $c \in \mathbb{M}/{\sim}$ or $c \in \mathbb{M}/{\sim^n}$, respectively. Note that $V_a^{\sim}$ refines $V_a^{\sim^n}$ for every $n$, and that all of these in particular encode the restriction of the original propositional assignment in $\mathbb{M}$.[5]

An information state $s \subseteq [w]_a$ of $\mathbb{M}_a(w)$ is called *colour-saturated* if it is a union of colour classes. The colour-saturation of a state $s \in \Sigma_a(w)$ is the set

$$\hat{s} = \bigcup \{ c \cap [w]_a \colon c \cap s \neq \emptyset \}.$$

**Definition 27.** A local $a$-structure $\mathbb{M}_a^{\sim}(w)$ or $\mathbb{M}_a^{\sim^n}(w)$ is *simple* if, for every $s \in \Sigma_a(w)$, also $\hat{s} \in \Sigma_a(w)$. In other words, if $\Sigma_a(w)$ is the downward closure of a collection of colour-saturated subsets of $[w]_a$. An inquisitive epistemic model $\mathbb{M}$ is *simple* if each one of its local $a$-structures $\mathbb{M}_a^{\sim}(w)$ is.[6]

Note that for a simple local structure, the propositional assignment fully determines the inquisitive assignment. Interestingly, $\sim$, as an equivalence on information states, is sufficiently flexible to always allow passage to simple models. This phenomenon also illustrates how little the actual relational encodings (the choice for the second sort) seems to interfere with the semantics of INQML. One can simply add to any $\Sigma_a(w)$ the colour-saturations of all $s \in \Sigma_a(w)$ together with all their subsets: clearly this process does not introduce any new bisimulation types of information states, and thus it yields a model which is globally bisimilar to the original one.

[5] This refers to a 'refinement' in terms of the induced partitions.

[6] Since $\sim^n$-classes are unions of $\sim$-classes, this implies that all the local structures $\mathbb{M}_a^{\sim^n}(w)$ are also simple.

**Lemma 28.** *Any inquisitive epistemic model is globally bisimilar to a simple one, which is based on the same set of worlds and induces the same modal $S5$ Kripke frame.*

**Definition 29.** A local $a$-structure $\mathbb{M}_a^{\sim}(w)$ or $\mathbb{M}_a^{\sim^n}(w)$ is $K$-*rich* for some $K \in \mathbb{N}$ if, for every information state $s \in \Sigma_a(w)$ there is some $s'$ such that $s \subseteq s' \in \Sigma_\alpha(w)$ in which colours of worlds occur with multiplicity at least $K$: $|c \cap s'| \geqslant K$ unless $c \cap s' = \emptyset$. An inquisitive epistemic model $\mathbb{M}$ is $K$-*rich* if each one of its local $a$-structures $\mathbb{M}_a^{\sim}(w)$ (and hence also each $\mathbb{M}_a^{\sim^n}(w)$) is $K$-rich.

**Lemma 30.** *For any $K \in \mathbb{N}$, any inquisitive epistemic model is globally bisimilar to a $K$-rich one based on a natural direct product with a $K$-clique, which preserves simplicity as well as finiteness.*

The following simple observation concerns a collapse phenomenon for local $a$-structures, which is familiar from standard mono-modal $S5$.

**Observation 31.** *Over single-agent inquisitive epistemic models, $\sim$ coincides with $\sim^1$.*

All the above notions analogously apply to any relational encodings of inquisitive epistemic models. For instance we refer to the local $a$-structure $\mathfrak{M}_a^{\sim}(w)$ induced by the relational encoding $\mathfrak{M} = \langle W, S, (E_a), (P_i), V \rangle$ as a relational inquisitive model with set of worlds $[w]_a \subseteq W$, set of information states $S \cap \wp([w]_a)$, single agent $a$ and atomic propositions in $W/\sim$ mapped to the corresponding subsets of $[w]_a$. Note that the second sort in this relational encoding depends on the choice of relational encoding $\mathfrak{M}$ for $\mathbb{M}$: in particular, the local structure $\mathfrak{M}_a^{\sim}(w)$ is a full relational model if $\mathfrak{M}$ itself is locally full.

The following lemma holds the key to the local component of an upgrading in the spirit of Figure 1, for relational encodings of sufficiently rich and simple local $a$-structures. Technically, it relies on an Ehrenfeucht–Fraïssé game argument for monadic second-order logic MSO over the first sort in order to deal with the information states in the second sort.

**Lemma 32.** *For any two simple local $a$-structures $\mathbb{M}_a^{\sim}(w)$ and $\mathbb{M}_a'^{\sim}(w)$ with locally full relational encodings $\mathfrak{M}^{\mathsf{lf}}(\mathbb{M}_a^{\sim}(w))$ and $\mathfrak{M}^{\mathsf{lf}}(\mathbb{M}_a'^{\sim}(w))$: if $\mathbb{M}_a^{\sim}(w)$ and $\mathbb{M}_a'^{\sim}(w)$ are sufficiently rich in relation to q, then $\sim^1$ coincides with $\equiv_q$ for any pair of worlds from $[w]_a \times [w']_a$.*

*The same claim holds w.r.t. the coarser local $a$-structures $\mathbb{M}_a^{\sim^n}(w)$ and $\mathbb{M}_a'^{\sim^n}(w)$; moreover, the claim for $\mathbb{M}_a^{\sim^n}(w)$ and $\mathbb{M}_a'^{\sim^n}(w)$ implies the claim for $\mathbb{M}_a^{\sim^m}(w)$ and $\mathbb{M}_a'^{\sim^m}(w)$ for any $m \leqslant n$.*

*Proofsketch.* The core argument is based on a representation of the constant and simple inquisitive assignments $\Sigma_a(w)$ and $\Sigma_a'(w')$ by their maximal and saturated members within the second sort, $\wp([w]_a)$ and $\wp([w']_a)$. By $\sim^1$-equivalence, the universes of the second sort are partitioned in a matching manner into colour classes. These colour classes, together with the maximal elements in $\Sigma_a(w)$ and $\Sigma_a'(w')$, which by simplicity are unions of colour classes, form the relevant tuples of parameters in the second sort, which need to be respected in the two-sorted FO Ehrenfeucht–Fraïssé game. Provided these colour classes are large enough, viz. of size exponential in the number of rounds $q$, this entails a winning strategy for the second player in the FO Ehrenfeucht–Fraïssé game over the relational encodings of the second sorts. Equivalently, by a folklore result stated as Observation 40 in the appendix, **II** has a winning strategy in the $q$-round MSO game over the first sort from any position in which all boolean combinations of set-parameters in the initial position match in being either trivial or of size at least $2^q$. In our case, this can be guaranteed by $K$-richness for sufficiently large $K$. Finally, simplicity and $K$-richness are preserved in a passage to coarser colour classes corresponding to coarser levels $\sim^n$. $\square$

Putting these results together, we already know that any two (finite) inquisitive epistemic models admit (finite) $\sim$-equivalent companions, which are simple and sufficiently rich, so that the locally full relational encodings are such that $\sim^1$-matching worlds are part of $\equiv_q$-equivalent local $a$-structures, for every $a \in \mathcal{A}$. It remains to pre-process the global pattern of connections between local $a$-structures in $\sim^n$-equivalent models to allow us to piece together these local $\equiv_q$-equivalences so as to support global $\equiv_q$-equivalence.

The global pattern of connections and overlaps between the constituent local $a$-structures is in fact governed by the underlying Kripke frames, and we correspondingly look for a process of partial unfolding for these, similar to techniques employed in [9] for standard modal logic over multi-modal $S5$ frames.[7]

### *B. Macroscopic upgrading through partial unfolding*

Recall from Section VI that, for our relational inquisitive epistemic models, Gaifman distance is graph distance in the undirected, bi-partite graph on the sets $W$ of worlds and $S$ of states with edges between any pair linked by the $E_a$ or by $\varepsilon$. On worlds, it thus corresponds to up to twice the distance induced by the $R_a$ on the underlying Kripke structures, which puts two distinct worlds $u, v$ at distance 1 iff $(u, v) \in R_a$ for some $a \in \mathcal{A}$. In other words, within a constant factor, Gaifman distance in the relational inquisitive epistemic frame corresponds to Gaifman distance in the underlying $S5$ Kripke frame. Also recall that the *$\ell$-neighbourhood* $N^\ell(w)$ of a world $w$ consists of all worlds or states at distance up to $\ell$ from $w$ in this bi-partite graph sense, and similarly for $N^\ell(s)$. The *$\ell$-localisation* of a formula $\varphi(x) \in \mathsf{FO}$ is its (syntactic) relativisation to the $\ell$-neighbourhood $N^\ell(x)$, denoted $\varphi^\ell(x)$. A tuple of worlds and/or states in a relational model is *$\ell$-scattered* if the $\ell$-neighbourhoods of its components are pairwise disjoint. Writing $\varphi^\ell(\mathfrak{M})$ for the set of worlds or states in $\mathfrak{M}$ that satisfy $\varphi^\ell(x)$, we say that $\varphi$ *is $(\ell, m)$-scattered in* $\mathfrak{M}$ if $\varphi^\ell(\mathfrak{M})$ contains an $\ell$-scattered $m$-tuple of elements.

[7]Note that a partial unfolding by stratification in the sense of Section VI-A is not compatible with the constraints on inquisitive epistemic frames.

**Definition 33.** Two pointed relational structures $\mathfrak{M}, w$ and $\mathfrak{M}', w'$ are *Gaifman* $(\ell, q, m)$ *equivalent*, $\mathfrak{M}, w \equiv_{q,m}^{(\ell)} \mathfrak{M}', w'$, if

(i) $w$ and $w'$ satisfy exactly the same $\ell$-localisations $\varphi^\ell(x)$ of formulae $\varphi(x) \in \mathsf{FO}_q$, and

(ii) for all $\ell_0 \leqslant \ell$, $m_0 \leqslant m$ and $\varphi(x) \in \mathsf{FO}_q$: $\varphi$ is $(\ell_0, m_0)$-scattered in $\mathfrak{M}$ if it is $(\ell_0, m_0)$-scattered in $\mathfrak{M}'$.

We also write $\mathfrak{M}, w \equiv_q^{(\ell)} \mathfrak{M}', w'$ for $\mathfrak{M}, w \equiv_{q,0}^{(\ell)} \mathfrak{M}', w'$ (which is the same as $\equiv_q$ of the pointed $\ell$-neighbourhoods).

Gaifman's theorem says that, for finite relational vocabularies, the two hierarchies of equivalences $\equiv_q$ and $\equiv_{q,m}^{(\ell)}$ are commensurate. In particular, any $\varphi(x) \in \mathsf{FO}$ is also preserved by sufficiently high levels of Gaifman equivalence [10].

**Theorem 34** (Gaifman). *Any first-order formula $\varphi(x)$ in a (finite) relational vocabulary is preserved under $\equiv_{q,m}^{(\ell)}$ for sufficiently high values of $\ell, q$, and $m$.*

Since we are interested in $\varphi(x) \in \mathsf{FO}$ that are $\sim$-invariant, and hence in particular under disjoint unions of models, the levels of Gaifman equivalence that matter to us are the simpler, purely local levels $\equiv_q^{(\ell)}$. Recall that, due to extensionality and the special rôle of the empty information state, we need to make do with 'essentially disjoint' compositions rather than actual disjoint unions of relational models. These compositions are based on the disjoint union of the sets of worlds and the almost disjoint union of the sets of information states—disjoint with the exception of the empty information state, which by extensionality is shared between all parts of any model. Writing again $\oplus_{i\in I}\mathfrak{M}_i$ for such an essentially disjoint sum of relational inquisitive models $\mathfrak{M}_i$ with shared information state $\emptyset$, preservation of $\varphi(x)$ under essentially disjoint sums means that, for any world $w$ of any one of the $\mathfrak{M}_i$,

$$\oplus_{i\in I}\mathfrak{M}_i, w \models \varphi \iff \mathfrak{M}_i, w \models \varphi,$$

which is in particular the case if $\varphi(x)$ is preserved under $\sim$, since obviously $\mathfrak{M}_i, w \sim \oplus_{i\in I}\mathfrak{M}_i$. Writing $\dot{\bigcup}_{i\in I}\mathfrak{M}_i$ for the actual disjoint union, which is not officially a relational inquisitive model, it is clear that there are simple translations between the FO-theories of $\dot{\bigcup}_{i\in I}\mathfrak{M}_i, w$ and of $\oplus_{i\in I}\mathfrak{M}_i, w$. (Technically, these structures are FO-inter-definable.) So every $\varphi(x) \in \mathsf{FO}$ that is preserved under under $\sim$, and thus also under essentially disjoint sums, translates into some $\varphi'(x) \in \mathsf{FO}$ that is preserved under actual disjoint unions and says that $\varphi(x)$ holds true of the models obtained by $\emptyset$-identification.

The following corollary to Theorem 34 shows that an upgrading argument in the spirit of Figure 1 can work with target levels $\equiv_q^{(\ell)}$.

**Corollary 35.** *Any first-order formula $\varphi(x)$ that is preserved under finite disjoint unions over a class $\mathcal{C}$ of relational structures that is closed under disjoint unions, is also preserved under $\equiv_q^{(\ell)}$ for some $\ell$ and $q$ over $\mathcal{C}$. The analogous assertion also holds for any $\varphi(x) \in \mathsf{FO}$ that is preserved under finite essentially disjoint sums over classes of relational inquisitive models with the analogous closure property.*

*Proof.* Let $\varphi(x)$ be preserved under $\equiv_{q,m}^{(\ell)}$ for suitable $\ell, q, m$ according to Theorem 34 and under disjoint unions. It suffices to show that whenever $\mathfrak{M} \restriction N^\ell(w), w \equiv_q \mathfrak{M}' \restriction N^\ell(w'), w'$, then $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M}', w' \models \varphi$. But $\mathfrak{M} \restriction N^\ell(w), w \equiv_q \mathfrak{M}' \restriction N^\ell(w'), w'$ implies that the disjoint sums of each of $\mathfrak{M}, w$ and $\mathfrak{M}', w'$ with $m$ disjoint isomorphic copies of both $\mathfrak{M}$ and $\mathfrak{M}'$, are equivalent in the sense of $\equiv_{q,m}^{(\ell)}$. By preservation of $\varphi$ under $\equiv_{q,m}^{(\ell)}$ and disjoint unions, $\mathfrak{M}, w \models \varphi \Leftrightarrow \mathfrak{M}', w' \models \varphi$ follows. □

In preparation for an upgrading according to Figure 3, we now want to smooth out obstructions to $\ell$-local equivalence $\equiv_q^{(\ell)}$ in locally full relational encodings of bisimilar companions $\mathbb{M}^*, w^* \sim \mathbb{M}, w$ and $\mathbb{M}'^*, w'^* \sim \mathbb{M}', w'$ of given inquisitive epistemic models $\mathbb{M}, w \sim^n \mathbb{M}', w'$ that are $n$-bisimilar for some suitable level $n = n(\ell, q)$. For this we aim at a locally tree-like structure for the underlying $S5$ Kripke frames, in bisimilar companions that allow us to import $\equiv_q$-equivalent second sorts with inquisitive assignments into the local $a$-structures over the equivalence classes $[w]_a$ (cf. Lemma 32). To this end, we want the overlaps between distinct equivalence classes $[w]_a$ to be as simple as possible and not to form any short cycles. These properties concern just the underlying $S5$ Kripke structure $\mathfrak{K}(\mathbb{M})$.

**Definition 36.** An $S5$ Kripke structure is $N$-acyclic if

(i) any two equivalence classes $[w]_a$ and $[w']_{a'}$ for distinct $a, a' \in \mathcal{A}$ intersect in at most one world;

(ii) any cyclic tuple of non-trivially[8] overlapping equivalence classes $([w_i]_{a_i})_{i\in\mathbb{Z}_n}$ has length $n > N$.

We correspondingly seek suitable bisimilar companions, firstly at the level of the underlying $S5$ Kripke structures w.r.t. ordinary modal bisimulation.

**Definition 37.** A map $\pi\colon \mathfrak{K}^* \xrightarrow{\sim} \mathfrak{K}$ between $S5$ Kripke structures is a *simple bisimilar covering* if

(i) the graph of $\pi$ induces a global bisimulation between $\mathfrak{K}^*$ and $\mathfrak{K}$ (i.e., $\pi$ is a bounded morphism);

(ii) each $R_a^*$ equivalence class $[w^*]_a$ of $\mathfrak{K}^*$ is bijectively mapped by $\pi$ onto the equivalence class $[\pi(w^*)]_a$ of $\mathfrak{K}$;

(iii) $\pi$ is finite-to-one, i.e. all pre-images $\pi^{-1}(w)$ are finite.

The following result from [9] can be obtained using natural bipartite graph representations of $S5$ structures in combination with products with suitable finite Cayley graphs of large girth.

**Proposition 38** ([9]). *Any $S5$ Kripke structure $\mathfrak{K}$ admits simple bisimilar coverings $\pi\colon \mathfrak{K}^* \xrightarrow{\sim} \mathfrak{K}$ by $N$-acyclic $S5$-structures $\mathfrak{K}^*$, for any given threshold $N$.*

### C. Proof of Theorem 2

It remains to combine the simple global bisimilar coverings of Proposition 38 with the passage to rich and simple local structure as discussed in Lemmas 30 and 32, in order to establish the crucial compactness property of Observation 23 via the upgrading in Figure 3.

[8] I.e., with $[w_i]_{a_i} \cap [w_{i+1}]_{a_{i+1}} \neq \emptyset$ and $[w_i]_{a_i} \neq [w_{i+1}]_{a_{i+1}}$.

$$\begin{array}{ccc}
\mathbb{M}, w & \text{——} \sim^n \text{——} & \mathbb{M}', w' \\
| & & | \\
\sim & & \sim \\
| & & | \\
\mathbb{M}_0, w_0 & \text{——} \sim^n \text{——} & \mathbb{M}_0', w_0' \\
| & & | \\
\sim & & \sim \\
| & & | \\
\mathbb{M}^*, w^* & \text{——} \sim^n \text{——} & \mathbb{M}'^*, w'^* \\
\vdots & & \vdots \\
\mathfrak{M}^*, w^* & \text{——} \equiv_q^{(\ell)} \text{——} & \mathfrak{M}'^*, w'^*
\end{array}$$

Fig. 3. Upgrading pattern for Theorem 2.

For this we pass from $\mathbb{M}, w \sim^n \mathbb{M}', w'$ firstly to simple and $K$-rich bisimilar companions $\mathbb{M}_0, w_0 \sim \mathbb{M}, w$ and $\mathbb{M}_0', w_0' \sim \mathbb{M}', w'$ according to Lemma 30, for $K$ large enough to support the claim of Lemma 32 for the target level $\equiv_q$ for which $\varphi$ will be preserved by $\equiv_q^{(\ell)}$ for appropriate $\ell$. We also choose $n \geqslant \ell/2+1$. Note that $n, \ell$ and $q$ are determined by $\varphi$, but we are here not concerned about the actual values. For the finite model theory version of Theorem 2 we also note that $\mathbb{M}_0$ and $\mathbb{M}_0'$ according to Lemma 30 are finite if $\mathbb{M}$ and $\mathbb{M}'$ are.

We next obtain $\mathbb{M}^*, w^* \sim \mathbb{M}_0, w_0$ from a simple bisimilar covering $\pi\colon \mathfrak{K}^* \xrightarrow{\sim} \mathfrak{K}$ of $\mathfrak{K} = \mathfrak{K}(\mathbb{M}_0)$ by an $N$-acyclic $S5$-structure $\mathfrak{K}^*$ for $N > \ell$ (cf. Proposition 38). In order to expand the $S5$ Kripke structure into an inquisitive epistemic model $\mathbb{M}^*$, we use the fact that simple bisimilar coverings induce bijections between $a$-classes in the covering structure and $a$-classes in the base structure: this mean that we can lift the (simple and $K$-rich) inquisitive assignments within the $a$-classes of $\mathbb{M}_0$, which are the $a$-classes of $\mathfrak{K}(\mathbb{M}_0)$, to their copies in the covering $\mathbb{M}^*$ so that $\pi \circ \Sigma_a^* := \Sigma_a \circ \pi$. Finally $w^*$ is chosen in $\pi^{-1}(w_0)$. It is easy to check that this process extends the $\pi$-induced modal bisimulation between the Kripke structures to a bisimulation of the inquisitive models. Moreover, the resulting inquisitive epistemic model $\mathbb{M}^*$ is finite if $\mathbb{M}_0$ is finite, and preserves the simplicity and richness properties of $\mathbb{M}_0$. The matching $\mathbb{M}'^*, w'^* \sim \mathbb{M}_0', w_0'$ and auxiliary $\mathfrak{K}'^*$ are similarly obtained.

Passage to locally full relational encodings $\mathfrak{M}^* := \mathfrak{M}^{\mathsf{lf}}(\mathbb{M}^*)$ and $\mathfrak{M}'^* := \mathfrak{M}^{\mathsf{lf}}(\mathbb{M}'^*)$ now leads to relational models, in which the $\ell$-neighbourhoods of the distinguished worlds $w^*$ and $w'^*$ are first-order equivalent up to quantifier rank $q$, i.e.,

$$\mathfrak{M}^*, w^* \equiv_q^{(\ell)} \mathfrak{M}'^*, w'^* \qquad (*)$$

so that $\mathfrak{M}^*, w^* \models \varphi$ iff $\mathfrak{M}'^*, w'^* \models \varphi$, as desired. The equivalence $(*)$ is established by means of a game argument: a strategy for the second player in the $q$-round first-order Ehrenfeucht–Fraïssé game on these two-sorted relational structures can be pieced together from

- (macro-level:) a strategy for the corresponding game on the locally tree-like $S5$ Kripke structures $\mathfrak{K}^*$ and $\mathfrak{K}'^*$ over the first sorts of $\mathfrak{M}^*$ and $\mathfrak{M}'^*$, and
- (micro-level:) strategies for the first-order game on the information states between local $a$-structures $(\mathfrak{M}^*)_a^{\sim^n}(u)$ and $(\mathfrak{M}'^*)_a^{\sim^n}(u')$ for $u$ and $u'$ matched by the strategy on worlds.

Some more detailed account of the game arguments derived from this two-level approach are given in the appendix, Section VIII-E.

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## VIII. APPENDIX

### A. Inquisitive Ehrenfeucht–Fraïssé: proof of the theorem

*Proof.* In this appendix we fill in the missing part in the proof of Theorem 15, showing that the characteristic formulae, as defined in connection with Proposition 16, have the following properties:

1) $\mathbb{M}', w' \models \chi^n_{\mathbb{M},w} \iff \mathbb{M}', w' \sim^n \mathbb{M}, w$
2) $\mathbb{M}', s' \models \chi^n_{\mathbb{M},s} \iff \mathbb{M}', s' \sim^n \mathbb{M}, t$ for some $t \subseteq s$
3) $\mathbb{M}', s' \models \chi^n_{\mathbb{M},\Pi} \iff \mathbb{M}', s' \sim^n \mathbb{M}, s$ for some $s \in \Pi$

First let us show that, if claim (1) holds for a certain $n \in \mathbb{N}$, then the claims (2) and (3) hold for $n$ as well.

For claim (2), suppose $\mathbb{M}', s' \models \chi^n_{\mathbb{M},s}$, that is, suppose $\mathbb{M}', s' \models \bigvee\{\chi^n_{\mathbb{M},w} \mid w \in s\}$. This requires that for any $w' \in s'$ we have $\mathbb{M}', w' \models \chi^n_{\mathbb{M},w}$ for some $w \in s$. By (1), this means that any world in $s'$ is $n$-bisimilar to some world in $s$. Letting $t$ be the set of worlds in $s$ that are $n$-bisimilar to some world in $s'$, we have $t \subseteq s$ and $\mathbb{M}', s' \sim_n \mathbb{M}, t$. Conversely, suppose $\mathbb{M}', s' \sim_n \mathbb{M}, t$ for some $t \subseteq s$. Then every $w' \in s'$ is $n$-bisimilar to some $w \in s$. By (1), this means that $\mathbb{M}', w' \models \chi^n_{\mathbb{M},w}$, which implies $\mathbb{M}', w' \models \chi^n_{\mathbb{M},s}$. Since this holds for any $w' \in s'$, and since $\chi^n_{\mathbb{M},s}$ is a truth-conditional formula (by Proposition 11), it follows that $\mathbb{M}', s' \models \chi^n_{\mathbb{M},s}$.

For claim (3), suppose that $\mathbb{M}', s' \models \chi^n_{\mathbb{M},\Pi}$. This requires $\mathbb{M}', s' \models \chi^n_{\mathbb{M},s}$ for some $s \in \Pi$. By (2) we have $\mathbb{M}', s' \sim_n \mathbb{M}, t$ for some $t \subseteq s$. Since $\Pi$ is downward closed, $t \in \Pi$. Conversely, suppose $\mathbb{M}', s' \sim_n \mathbb{M}, t$ for some $t \in \Pi$. By (2), $\mathbb{M}', s' \models \chi^n_{\mathbb{M},t}$, and since $t \in \Pi$, also $\mathbb{M}', s' \models \chi^n_{\mathbb{M},\Pi}$.

Next, we use these results to show that claim (1) holds for all $n \in \mathbb{N}$, by induction on $n$. The claim $\mathbb{M}', w' \models \chi^0_{\mathbb{M},w} \iff \mathbb{M}', w' \sim_0 \mathbb{M}, w$ follows immediately from the definition of $\chi^0_{\mathbb{M},w}$. Now assume that claim (1), and thus also claims (2) and (3), hold for $n$, and let us consider the claim for $n+1$.

For the right-to-left direction, suppose $\mathbb{M}', w' \sim_{n+1} \mathbb{M}, w$. We want to show that $\mathbb{M}', w' \models \chi^{n+1}_{\mathbb{M},w}$. This amounts to showing that: (i) $\mathbb{M}', w' \models \chi^n_{\mathbb{M},w}$; (ii) $\mathbb{M}', w' \models \boxplus\chi^n_{\mathbb{M},\Sigma(w)}$; and (iii) $\mathbb{M}', w' \models \neg\boxplus\chi^n_{\mathbb{M},\Pi}$ when $\Pi \subseteq \Sigma(w)$ and $\Pi \not\sim_n \Sigma(w)$. Let us show each in turn.

(i) $\mathbb{M}', w' \sim_{n+1} \mathbb{M}, w$ implies $\mathbb{M}', w' \sim_n \mathbb{M}, w$, so by the induction hypothesis $\mathbb{M}', w' \models \chi^n_{\mathbb{M},w}$.

(ii) Take $s' \in \Sigma'(w')$. Since $\mathbb{M}', w' \sim_{n+1} \mathbb{M}, w$ we have $\mathbb{M}', s' \sim_n \mathbb{M}, s$ for some $s \in \Sigma(w)$. By the induction hypothesis, it follows that $\mathbb{M}', s' \models \chi^n_{\mathbb{M},\Sigma(w)}$. Since this is true of all $s' \in \Sigma'(w')$, we have $\mathbb{M}', w' \models \boxplus\chi^n_{\mathbb{M},\Sigma(w)}$.

(iii) Suppose for a contradiction that for some $\Pi \subseteq \Sigma(w)$, $\Pi \not\sim_n \Sigma(w)$ and $\mathbb{M}', w' \models \boxplus\chi^n_{\mathbb{M},\Pi}$. This means that every $s' \in \Sigma'(w')$ supports $\chi^n_{\mathbb{M},\Pi}$ and thus, by our induction hypothesis, is $n$-bisimilar to some $s \in \Pi$. Since $\Pi \subseteq \Sigma(w)$ and $\Pi \not\sim_n \Sigma(w)$, there must be a state $t \in \Sigma(w)$ which is not $n$-bisimilar to any $s \in \Pi$. But since any state $s' \in \Sigma'(w')$ is $n$-bisimilar to some $s \in \Pi$, this means that $t$ is not $n$-bisimilar to any $s' \in \Sigma'(w')$. Since $t \in \Sigma(w)$, this contradicts the assumption that $\mathbb{M}', w' \sim_{n+1} \mathbb{M}, w$.

This establishes the right-to-left direction of the claim. For the converse, suppose $\mathbb{M}', w' \models \chi^{n+1}_{\mathbb{M},w}$. To prove that $\mathbb{M}', w' \sim_{n+1} \mathbb{M}, w$, we must show three things: (i) $w'$ and $w$ coincide on atomic formulae (ii) any state in $\Sigma'(w')$ is $n$-bisimilar to some state in $\Sigma(w)$; and (iii) any state in $\Sigma(w)$ is $n$-bisimilar to some state in $\Sigma'(w')$. Let us show each in turn.

(i) Since $\chi^n_{\mathbb{M},w}$ is a conjunct of $\chi^{n+1}_{\mathbb{M},w}$, by the induction hypothesis we have $\mathbb{M}', w' \sim_n \mathbb{M}, w$, which implies that $w$ and $w'$ make true the same atomic formulae.

(ii) Since $\boxplus\chi^n_{\mathbb{M},\Sigma(w)}$ is a conjunct of $\chi^{n+1}_{\mathbb{M},w}$ we have $\mathbb{M}', w' \models \boxplus\chi^n_{\mathbb{M},\Sigma(w)}$. This implies that any $s' \in \Sigma'(w')$ supports $\chi^n_{\mathbb{M},\Sigma(w)}$. By the induction hypothesis, this means that any $s' \in \Sigma'(w')$ is $n$-bisimilar to some $s \in \Sigma(w)$.

(iii) Let $\Pi$ be the set of states in $\Sigma(w)$ which are $n$-bisimilar to some $s' \in \Sigma'(w')$. Now, consider any $s' \in \Sigma'(w')$. We have already seen that $s'$ is $n$-bisimilar to some state $s \in \Sigma(w)$, which must then be in $\Pi$ by definition. By induction hypothesis, the fact that $s'$ is $n$-bisimilar to some state in $\Pi$ implies $\mathbb{M}', s' \models \chi^n_{\mathbb{M},\Pi}$. And since this is true for each $s' \in \Sigma'(w')$, we have $\mathbb{M}', w' \models \boxplus\chi^n_{\mathbb{M},\Pi}$. Now suppose towards a contradiction that some state $s \in \Sigma(w)$ were not $n$-bisimilar to any state in $\Sigma'(w')$. Then, $s$ would not be $n$-bisimilar to any state in $\Pi$ either. This would mean that $\Pi \not\sim_n \Sigma(w)$, which means that $\neg\boxplus\chi^n_{\mathbb{M},\Pi}$ is a conjunct of $\chi^{n+1}_{\mathbb{M},w}$. But then, since $\mathbb{M}', w' \models \chi^{n+1}_{\mathbb{M},w}$, we should have $\mathbb{M}', w' \models \neg\boxplus\chi^n_{\mathbb{M},\Pi}$, which contradicts our conclusion above.

This completes the proof of Proposition 16. We can then use the properties of our characteristic formulae to prove the non-trivial direction of Theorem 15. For suppose $\mathbb{M}, s \not\sim_n \mathbb{M}', s'$: then either of the states $s$ and $s'$ is not $n$-bisimilar to any subset of the other. Without loss of generality, say it is $s'$. By the property of the formula $\chi^n_{\mathbb{M},s}$ we have $\mathbb{M}, s \models \chi^n_{M,s}$ but $\mathbb{M}', s' \not\models \chi^n_{\mathbb{M},s}$. Since the modal depth of $\chi^n_{\mathbb{M},s}$ is $n$, this shows that $\mathbb{M}, s \not\equiv^n_{\text{INQML}} \mathbb{M}', s'$. □

### B. Failures of compactness over classes of relational models

Observation 24 refers to the following example of a first-order property of worlds $w$ in *full* relational models that is $\sim$-invariant and (as a well-foundedness assertion) obviously incompatible with compactness. In terms of the accessibility relation $R = \{(u,v) : \exists s \text{ s.t. } uEs \text{ and } v \in s\}$:

$$\mathbb{P}(w) := \text{ there is no infinite } R\text{-path from } w$$

Clearly this property $\mathbb{P}$ is not expressible in INQML: although it is $\sim$-invariant, it is clearly not invariant under any finite level of bisimulation equivalence. It is first-order definable over full relational encodings, because those afford the full expressive power of monadic second-order quantification over the first sort, $W$, via first-order quantification over the second sort $S = \wp(W)$. The following MSO-formula, which defines $\mathbb{P}$ over the underlying Kripke frame, can therefore be translated into first-order terms over full relational encodings:

$$\neg\exists X\big(x \in X \wedge \forall y\big(y \in X \to \exists z(z \in X \wedge Ryz)\big)\big).$$

This shows that the analogue of our Theorem 1 fails for the class of full relational models: over this class, there are properties that are FO–definable and $\sim$-invariant, but which are not definable in INQML. It is also possible to show that compactness fails over the class of all relational models (which is not an elementary class). However, in this case, Theorem 1 together with the compactness of INQML implies that this cannot happen for FO-formulae that are $\sim$-invariant.

The counterexample to compactness is obtained by relativising the property $\mathbb{P}$ given above to the subframe induced by any non-empty information state $s$ in the image of $\Sigma$: in restriction to $s \subseteq W$, any relational encoding of $\langle W, \Sigma \rangle$ provides the full power set $\wp(s)$ in the second sort.

**Example 39.** There is a first-order formula in a single free variable the second sort (information state) saying of non-empty information states $s \in S$ in the second sort of any relational inquisitive frame with first sort $W$ that there are no infinite $R$-paths included in $s$.

*Proof.* We simply relativise to $s$ the above formula that defines property $\mathbb{P}$, and then universally quantify over $w \in s$. $\square$

### C. Locality in stratified models: the game argument

The game argument at the heart of the proof of Theorem 1 in Section VI needs to establish

$$(**) \quad \mathfrak{M}_0, w \equiv_q \mathfrak{M}_1, w$$

for structures

$$\begin{array}{ll} \mathfrak{M}_0, w = & q \otimes \mathfrak{M} \;\oplus\; \mathfrak{M} \restriction N^\ell(w), w \;\oplus\; q \otimes \mathfrak{M} \restriction N^\ell(w) \\ \mathfrak{M}_1, w = & q \otimes \mathfrak{M} \;\oplus\; \qquad \mathfrak{M}, w \qquad \oplus\; q \otimes \mathfrak{M} \restriction N^\ell(w) \end{array}$$

obtained as essentially disjoint sums from a world-pointed relational inquisitive model $\mathfrak{M}, w$ that is stratified to depth $\ell = 2^q$, so that its truncation $\mathfrak{M} \restriction N^\ell(w)$ is stratified. 'Essentially disjoint sums' here refers to disjoint sums with identifications of the empty information states $\emptyset$ across the disjoint parts. It is not hard to see that the $q$-equivalence claim in $(**)$, however, is insensitive to whether the empty information states, which are uniformly present in the second sort of each component, are identified or not. So we may as well work with proper disjoint unions in the following proof.

*Proof of* $(**)$. We argue that the second player has a winning strategy in the classical $q$-round Ehrenfeucht–Fraïssé game over these two structures starting in the position with a single pebble on the distinguished world $w$ on either side. Indeed, the second player can force a win by maintaining the following invariant w.r.t. the game positions $(\mathbf{u}; \mathbf{u}')$ for $\mathbf{u} = (u_0, u_1, \ldots, u_m)$ with $u_0 = w$ in $\mathfrak{M}_0$ and $\mathbf{u}' = (u'_0, u'_1, \ldots, u'_m)$ with $u'_0 = w$ in $\mathfrak{M}_1$ after round $m$, for $m = 0, \ldots, q$, for $\ell_m := 2^{q-m}$:

> $\mathbf{u}$ and $\mathbf{u}'$ are partitioned into clusters of matching sub-tuples such that the distance between separate clusters is greater than $\ell_m$ and corresponding clusters are in isomorphic configurations of isomorphic component structures of $\mathfrak{M}_0$ and $\mathfrak{M}_1$ or in isomorphic configurations in $\mathfrak{M}_0 \restriction N^\ell(w)$ and $\mathfrak{M}_1 \restriction N^\ell(w)$.

This condition is satisfied at the start of the game, for $m = 0$. The second player can maintain this condition through a single round, say in the step from $m$ to $m+1$, as follows. Suppose the first player puts a pebble in position $u = u_{m+1}$ in $\mathfrak{M}_0$ or $u' = u'_{m+1}$ in $\mathfrak{M}_1$ at distance up to $\ell_{m+1}$ of one of the level $m$ clusters (it cannot fall within distance $\ell_{m+1}$ of two distinct clusters, since the distance between two distinct clusters from the previous level is greater than $\ell_m = 2\ell_{m+1}$); then this new position joins a sub-cluster of that cluster and its match is found in an isomorphic position relative to the matching cluster. If the first player puts the new pebble in a position $u = u_{m+1}$ in $\mathfrak{M}_0$ or $u' = u'_{m+1}$ in $\mathfrak{M}_1$ at distance greater than $\ell_{m+1}$ of each one of the level $m$ clusters, then this new position will form a new cluster of its own and can be matched with an isomorphic position in one of the as yet unused component structures on the opposite side.

All steps of this proof restrict naturally to the scenarios of (finite or general) locally full relational inquisitive structures. $\square$

### D. A simple Ehrenfeucht–Fraïssé observation about MSO

This observation is used in Section VII towards the proof of Lemma 32.

To compare sizes of sets up to a critical value $d$ (thinking of $d$ as a threshold beyond which precise distinctions cease to matter), we write $|s| =_d |s'|$ if $|s| = |s'|$ or $|s|, |s'| \geqslant d$. For tuples $\mathbf{s} = (s_1, \ldots, s_k)$ of subsets $s_i \subseteq W$ and $\mathbf{s}' = (s'_1, \ldots, s'_k)$ of subsets $s'_i \subseteq W'$, the equivalence

$$\mathbf{s} =_d \mathbf{s}'$$

means that $\zeta(\mathbf{s}) =_d \zeta(\mathbf{s}')$ for every boolean term $\zeta$ that describes a boolean combination of the $k$ sets in question. The following is then folklore; we just sketch the proof idea.[9]

**Observation 40.** *For tuples of subsets* $\mathbf{s} = (s_1, \ldots, s_k)$ *of* $W$ *and* $\mathbf{s}' = (s'_1, \ldots, s'_k)$ *of* $W'$*, and for* $d = 2^q$*:*

$$\mathbf{s} =_d \mathbf{s}' \;\;\Rightarrow\;\; (W, \mathbf{s}) \equiv_q^{\mathsf{MSO}} (W', \mathbf{s}').$$

*Proofsketch..* The proof is by induction on $q$. For the induction step assume that $\mathbf{s} =_{2d} \mathbf{s}'$ and suppose w.l.o.g. that player **I** proposes a subset $s \subseteq W$ so that **II** needs to find a matching $s' \subseteq W'$ such that $\mathbf{s}s =_d \mathbf{s}'s'$. Decompose $s$ and its complement $\bar{s}$ into their intersections with the atoms of the boolean algebra generated by $\mathbf{s}$ in $\wp(W)$. Then each part of these partitions of $s$ and $\bar{s}$ can be matched with a subset of the corresponding atom of the boolean algebra generated by $\mathbf{s}'$ in $\wp(W')$ in such a manner that the parts $\zeta(\mathbf{s}) \cap s$ and $\zeta(\mathbf{s}) \setminus s$ match their counterparts $\zeta(\mathbf{s}') \cap s'$ and $\zeta(\mathbf{s}') \setminus s'$ in the sense of $=_d$. This just uses the assumption that $|\zeta(\mathbf{s})| =_{2d} |\zeta(\mathbf{s}')|$. $\square$

### E. Upgrading for Theorem 2: the detailed game argument

We supply some more detail regarding the Ehrenfeucht–Fraïssé game argument at the heart of our proof of Theorem 2. On the basis of the development in Section VII, we still need

[9]The analysis naturally extends to infinite families of subsets, but we only encounter very tame instances of that, when dealing with infinite colour sets $C$ induced by infinitely many bisimulation types in $\mathbb{M}/{\sim}$ over infinite $\mathbb{M}$, in which case the local $a$-structures can be decomposed into their infinitely many disjoint colour classes for the corresponding local equivalence considerations.

to argue that the pre-processed, pointed structures $\mathfrak{M}^*, w^*$ and $\mathfrak{M}'^*, w'^*$ in Figure 3 are $\ell$-locally $q$-equivalent in the sense of $\equiv_q^{(\ell)}$, i.e., that **II** has a winning strategy from position $(w^*, w'^*)$ in the classical $q$-round Ehrenfeucht–Fraïssé game on $\mathfrak{M}^* \restriction N^\ell(w^*)$ versus $\mathfrak{M}'^* \restriction N^\ell(w'^*)$:

**Claim 41.** *In this situation $\mathfrak{M}^*, w^* \equiv_q^{(\ell)} \mathfrak{M}'^*, w'^*$.*

*Proof.* To ease notation, rename $\mathfrak{M}^* \restriction N^\ell(w^*), w^* =: \mathfrak{M}, w$ and $\mathfrak{M}'^* \restriction N^\ell(w'^*), w'^* =: \mathfrak{M}', w'$ and assume the results of the pre-processing for these. Specifically, this pre-processing guarantees that

(a) the underlying Kripke structures are $N$-acyclic for some $N > \ell$, so that their $\ell$-localisations $\mathfrak{K}$ and $\mathfrak{K}'$ are properly acyclic w.r.t. the overlap pattern between their $a$-classes and the induced local $a$-structures, which is tree-like of depth $d \leqslant \ell/2 \leqslant n-1$ with just singleton overlaps as links.

(b) the local $a$-structures w.r.t. colour classes that describe $\sim^n$-classes are such that $\sim^1$ between any pair of worlds implies $q$-equivalence in the two-sorted relational encodings of these local $a$-structures (cf. Lemma 32). For worlds $u$ and $u'$ of the same colour w.r.t. any $\sim^m$, $m \leqslant n$:

$$\mathfrak{M}_a^{\sim^m}(u), u \equiv_q \mathfrak{M}_a'^{\sim^m}(u'), u'.$$

Consider any position $\mathbf{u}; \mathbf{u}'$ in the $q$-round first-order game on $\mathfrak{M}$ and $\mathfrak{M}'$ starting with pebbles on $w$ and $w'$ which are equivalent w.r.t. $\sim^n$. It will be of the form $\mathbf{u} = (u_0, \dots, u_k)$ and $\mathbf{u}' = (u'_0, \dots, u'_k)$ for some $k \leqslant q$ with $u_0 = w$, $u'_0 = w'$, and, for $1 \leqslant i \leqslant k$, either

(i) $u_i = w_i$ and $u'_i = w'_i$ are elements of the first sort (worlds) that are linked to $w$ and $w'$, respectively, by unique paths of overlapping $a$-classes, or

(ii) $u_i = s_i$ and $u'_i = s'_i$ are of the second sort (information states) in some $\Sigma_a(w_i)$, respectively $\Sigma'_a(w'_i)$, for a pair of uniquely determined worlds $w_i$ and $w'_i$ as in (i).

We describe such positions after $k$ rounds, for $k \leqslant q$, in terms of the associated $\mathbf{w} = (w_0, \dots, w_k)$ and $\mathbf{w}' = (w'_0, \dots, w'_k)$ and (for some $i$) $s_i \in \Sigma_{a_i}(w_i)$ and $s'_i \in \Sigma_{a'_i}(w'_i)$, which we further augment as follows.

With the tuple $\mathbf{w}$ of worlds in $\mathfrak{M}$ we associate the tree-like hypergraph structure, whose hyperedges represent overlapping $a$-classes in the underlying basic modal $S5$-frame $\mathfrak{K}$, corresponding to the minimal spanning sub-tree containing the worlds in $\mathbf{w}$. We write $\mathrm{tree}(\mathbf{w})$ for this tree structure. Its vertices consist of the worlds in $\mathbf{w}$ and, for every $w_i$ in $\mathbf{w}$, the unique sequence of worlds in which the $a$-classes that make up the shortest connecting path from the root $w$ to $w_i$ intersect. Its hyperedges are the non-trivial subsets of vertices that fall within the same $a$-class, labelled or coloured by the corresponding $a$. We also colour each vertex $u$ of this tree structure by its $\sim^{n-d}$-type in the original model, for $d = d(w, u)$ (distance from the root $w = w_0$, or depth in the tree structure). The tree $\mathrm{tree}(\mathbf{w}')$ on the side of $\mathfrak{M}'$ is similarly defined. After $k$ rounds, each of these trees can have at most $1 + k\ell/2$ vertices and each individual hyperedge can have at most $k+1$ vertices. We argue that **II** can maintain the following conditions in terms of these tree structures through $q$ rounds, which then also guarantees her to win:

(1) the tree structures $\mathrm{tree}(\mathbf{w})$ and $\mathrm{tree}(\mathbf{w}')$ spanned by $\mathbf{w}$ and $\mathbf{w}'$ are isomorphic (as hyperedge- and vertex-coloured hypergraphs) via an isomorphism $\rho$ that maps $\mathbf{w}$ to $\mathbf{w}'$:

$$\rho\colon \mathrm{tree}(\mathbf{w}), \mathbf{w} \simeq \mathrm{tree}(\mathbf{w}'), \mathbf{w}'$$

(2) for each local $a$-structure on $[u]_a$ for any pair of vertices $u \in \mathrm{tree}(\mathbf{w})$ and $u' \in \mathrm{tree}(\mathbf{w}')$ that are related by the isomorphism $\rho$ from (1) and at depth $d = d(w, u) = d(w', u')$:

$$\mathfrak{M}_a^{\sim^m}(u), \mathbf{s} \equiv_{q-z+1} \mathfrak{M}_a'^{\sim^m}(u'), \mathbf{s}'$$

for $m = n - d - 1$, and where $\mathbf{s}$ and $\mathbf{s}'$ are tuples of size $z$ that coherently list any singleton information states corresponding to the tree vertices incident with that $a$-class and any information states $s_i \in \Sigma_a(u)$ and $s'_i \in \Sigma_a(u')$ that may have been chosen during the first $k$ rounds of the game.

Both conditions are satisfied at the start of the game: (2) in this case does not add anything beyond (1), which in turn is a consequence of the assumption that $w_0 = w$ and $w'_0 = w'$ satisfy the same $\sim^n$-type in the original models.

We show how to maintain conditions (1) and (2) through round $k$, in which **I** may either choose an information state ($s_k$ or $s'_k$) or a world ($w_k$ or $w'_k$). We refer to the position before this round as described by parameters $\mathbf{w}, \mathbf{w}', \mathrm{tree}(\mathbf{w}), \dots$ as above but at level $k-1$, and assume conditions (1) and (2) for those. The following shows how **II** can find responses and update the auxiliary information so as to maintain conditions (1) and (2). We note that the situation in entirely symmetric w.r.t. $\mathfrak{M}$ and $\mathfrak{M}'$ so that we may w.l.o.g. assume that **I** moves on the side of $\mathfrak{M}$.

Case 1. Suppose **I** chooses a non-empty, non-singleton information state $s_k \in \Sigma_a(w_k)$, where $w_k$ and $a$ are uniquely determined by $s_k$.

Case 1.1. If $w_k = u$ for some vertex $u$ in $\mathrm{tree}(\mathbf{w})$, $u$ at depth $d$ say, then we look at one round in the game for

$$\mathfrak{M}_a^{\sim^m}(u), \mathbf{s} \equiv_{q-z+1} \mathfrak{M}_a'^{\sim^m}(u'), \mathbf{s}'$$

for $m = n - d - 1$. In that game, a move by **I** on $s_k$, which extends $\mathbf{s}$ to $\mathbf{s}, s_k$, has an adequate response $s'_k$ for **II**, which extends $\mathbf{s}'$ to $\mathbf{s}s'_k$ and guarantees $\equiv_{q-z}$. This is the appropriate level of equivalence, since the tuples $\mathbf{s}$ and $\mathbf{s}'$ have been extended by one component.

Case 1.2. If $w_k$ is 'new', then the appropriate $w'_k$ that satisfies conditions (1) and (2) has to be located in a first step that simulates a move on $w_k$ (treated as Case 2), after which we may proceed as in Case 1.1.

Case 2. Suppose **I** chooses a world $w_k$. The choice of an appropriate matching choice for $w'_k$ is treated by induction on the distance that the newly chosen world $w_k$ has from $\mathrm{tree}(\mathbf{w})$. In the base case, distance 0 from $\mathrm{tree}(\mathbf{w})$, $w_k$ is a vertex of $\mathrm{tree}(\mathbf{w})$ nothing needs to be updated: the response is dictated by the existing isomorphism $\rho$ according to (1).

In all other cases, $\mathrm{tree}(\mathbf{w})$ and $\rho$ need to be extended to encompass the new $w_k$. The new world $w_k$ can be joined to $\mathrm{tree}(\mathbf{w})$ by a unique shortest path of overlapping $a$-classes of length greater than 0 that connects it to $\mathrm{tree}(\mathbf{w})$. The new branch in $\mathrm{tree}(\mathbf{w})$ will be joined to $\mathrm{tree}(\mathbf{w})$ either through a new $a$-hyperedge emanating from an existing vertex $u$ of $\mathrm{tree}(\mathbf{w})$ (see Case 2.1) or through a new vertex $u$ in a local $a$-structure corresponding to an existing hyperedge (see Case 2.2).

Case 2.1. It is instructive to look at the special case of distance 1 from $\mathrm{tree}(\mathbf{w})$ and then argue how to iterate for larger distances. So let $w_k$ be at distance 1 from $\mathrm{tree}(\mathbf{w})$ in the sense that $w_k \in [u]_a$ for some $u$ in $\mathrm{tree}(\mathbf{w})$ at depth $d$ that is not incident with an $a$-hyperedge in $\mathrm{tree}(\mathbf{w})$. Since $u$ and $u' = \rho(u)$ have matching $\sim^{n-d}$ colours and

$$\mathfrak{M}_a^{\sim^m}(u), u \equiv_q \mathfrak{M}_a'^{\sim^m}(u'), u'$$

for $m = n-d-1$, a suitable response to the move that pebbles the world $w_k$ (or the singleton information state $\{w_k\}$) in that game yields a world $w'_k \in [u']_a$ with a $\sim^{n-d-1}$-colour that matches $w_k$ and such that

$$\mathfrak{M}_a^{\sim^m}(u), \{u\}, \{w_k\} \equiv_{q-1} \mathfrak{M}_a'^{\sim^m}(u'), \{u'\}, \{w'_k\}.$$

These levels of colouring and equivalence are appropriate, since the depth of $w_k$ and $w'_k$ is $d+1$, and since one new vertex contributes to the new hyperedge. So we may extend the isomorphism $\rho$ to map $w_k$ to $w'_k$ in keeping with conditions (1) and (2). If $w_k$ is at greater distance from its nearest neighbour $u$ in $\mathrm{tree}(\mathbf{w})$ we can iterate this process of introducing new hyperedges with one new element at a time, degrading the level of inquisitive bisimulation equivalence by 1 in every step that takes us one step further away from the root of the trees, but maintaining equivalences $\equiv_{q-1}$ between the newly added local $a$-structures.

Case 2.2. It remains to argue for the case of $w_k \in [u]_a$ for some $u$ in $\mathrm{tree}(\mathbf{w})$ that is already incident with an $a$-hyperedge of $\mathrm{tree}(\mathbf{w})$. By (2) we have

$$\mathfrak{M}_a^{\sim^m}(u), \mathbf{s} \equiv_{q-z} \mathfrak{M}_a'^{\sim^m}(u'), \mathbf{s}',$$

for $m = n-d-1$, where $z$ is the size of the tuples $\mathbf{s}$ and $\mathbf{s}'$ already incident with these local $a$-structures and $d$ is the depth of $u$ and $u'$ in the trees. So we can find a response to a move on $\{w_k\}$ in this game that yields $w'_k \in [u']_a$ of matching $\sim^{n-d-1}$-colour and such that

$$\mathfrak{M}_a^{\sim^m}(u), \mathbf{s}, \{w_k\} \equiv_{q-z-1} \mathfrak{M}_a'^{\sim^m}(u'), \mathbf{s}', \{w'_k\}.$$

These are the appropriate levels of colouring and equivalence, since the depth of $w_k$ and $w'_k$ is one greater than that of $u$ and $u'$, and since the length of the tuples $\mathbf{s}$ and $\mathbf{s}'$ has been increased by 1.

So we may extend $\rho$ by matching $w_k$ with $w'_k$ and extending $\mathrm{tree}(\mathbf{w})$ and $\mathrm{tree}(\mathbf{w})'$ by these new vertices and stay consistent with conditions (1) and (2). $\square$