Inquisitive Semantics and Intermediate Logics.

MSc Thesis (*Afstudeerscriptie*)

written by

Ivano A. Ciardelli
(born February 3rd, 1985 in Sesto san Giovanni, Italy)

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Date of the public defense:	**Members of the Thesis Committee:**
September 3, 2009	Prof Dr Johan van Benthem
	Prof Dr Jeroen Groenendijk
	Dr Rosalie Iemhoff
	Prof Dr Dick de Jongh
	Dr Floris Roelofsen
	Dr Yde Venema

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

[. . .]
Sono allora un soldatino di piombo
appeso a un filo sul mare aperto
I miei pensieri passano come velieri
[. . .]

Leonardo Martellini, *Velieri*

To my schoolteacher Mauro, who taught me how to doubt

Contents

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Overview of the thesis

The present thesis is concerned with inquisitive semantics and its logic. Inquisitive semantics is a system (or rather a class of systems) for modelling the effect of utterances in a cooperative dialogue; the crucial feature of inquisitive semantics is that it regards propositions as real *proposals* of one or more possible updates for the common ground of a conversation, thus allowing for the representation of both inquisitive and informative content as two aspects of a unique notion of meaning. We start with a general overview of the structure and the contents of the thesis.

Chapter 1: an introduction to inquisitive semantics. We introduce the reader to the ideas that motivate and guide the development of an inquisitive semantics. We first mention a traditional approach to the modelling of the exchange of information in a dialogue. We then illustrate a different perspective on cooperative information exchange, namely the propositions-as-proposals view which informs the construction of inquisitive semantics. We conclude the chapter with some historical notes, clarifying how the present thesis relates to previous and ongoing work on the subject.

Chapter 2: propositional inquisitive semantics. We set out to implement the ideas expounded in the previous chapter for a propositional language. We evaluate formulas over sets of valuations, conceived of as *information states*. The semantics is based upon the relation of *support* between states and propositional formulas: via the notion of support, we associate each formula with a set of *possibilities*, defined as maximal states supporting the formula. This set of possibilities represents the proposal put forward by a formula.

We identify two effects of a proposal: an *informative* effect consisting in the suggestion to eliminate some possible worlds from the common ground, and an *inquisitive* effect consisting in the specification of alternative updates. We define *assertions* and *questions* as formulas that serve only one of these purposes and discuss the properties of these classes of formulas, in particular showing that each formula can always be decomposed into a question and an assertion, its assertive part coinciding with its classical meaning.

We discuss several other properties of the semantics, such as expressive completeness and normal form results, point out a tight connection linking inquisitive semantics to intuitionistic Kripke semantics, and suggest an intuitive interpretation of the notion of support as 'knowing *how*'.

Chapter 3: inquisitive logic. We investigate the logic InqL stemming from the semantics discussed in the previous chapter. We show that it is a sort of intermediate logic lacking the property of closure under uniform substitution: in particular, the double negation law holds for atoms but fails in general.

We first obtain a sound and complete axiomatization for InqL by expanding intuitionistic logic with the Kreisel-Putnam scheme $(\neg\chi \to \varphi \vee \psi) \to (\neg\chi \to$

$\varphi) \vee (\neg\chi \to \psi)$ and the double negation axiom $\neg\neg p \to p$ restricted to atoms: the completeness proof exploits canonical model techniques from the field of intermediate logics, relying on the connection between inquisitive semantics and intuitionistic Kripke models to transfer counterexamples into the inquisitive setting.

We then develop some machinery to deal with non-substitution closed intermediate logics and show how to obtain a purely syntactic completeness proof exploiting the fact that, in InqL, any formula can be represented as a disjunction of negations. This proof is not only speedier and more transparent, but also more general, yielding a whole range of intermediate logics that, when expanded with the atomic double negation axiom, axiomatize InqL.

Using the same methods we also prove that, just as classical logic coincides with the negative fragment of intuitionistic logic, so inquisitive logic coincides with the disjunctive-negative fragment, that is, the fragment consisting of disjunctions of negations.

We also show that the schematic fragment of InqL –the set of formulas that are schematically valid– coincides with Medvedev's logic ML of finite problems. This establishes an interesting link between derivability in ML and derivability in other, better understood logics such as the Kreisel-Putnam logic; furthermore, it allows us to give an *exact* characterization of the intermediate logics that axiomatize InqL when expanded with atomic double negation: these are precisely the logics included between a weak variant of the Kreisel-Putnam logic and Medvedev's logic.

Finally, the connection with ML is used to establish the independence of the connectives in inquisitive semantics.

Chapter 4: the inquisitive hierarchy. In the original implementation of inquisitive semantics, formulas were evaluated not on states but on ordered pairs of valuations. We show that that semantics is obtained as a special case of the semantics discussed in this thesis when we restrict ourselves to states of cardinality at most two.

Thus, the 'pair' semantics can be regarded as an element of a sequence of semantics obtained by restricting the 'generalized' semantics to states of size an most n. We show how to extend our axiomatization of InqL to obtain axiomatizations of the logics connected to the layers of this hierarchy of semantics.

We conclude showing that all of the restricted semantics fail to yield the intended notion of possibilities and arguing in favour of the generalized system.

Chapter 5: intermediate logics with negative atoms. We take a more general perspective on the following problem we encountered in chapter 3: if we expand an intermediate logic Λ with the double negation axiom $\neg\neg p \to p$ restricted to atoms, do we create new schematic validities, and if so, which ones?

We show that the operation of taking the schematic validities of the expanded logic is a closure operator on the lattice of intermediate logic, which additionally preserves the disjunction property. We solve the mentioned problem for the best-known intermediate logics, finding that many of them are in fact stable under this 'negative closure' operation.

Chapter 6: first-order inquisitive semantics. We turn to the problem of setting up inquisitive semantics for a first-order language. Information states will now be sets of first-order models, over a domain that we assume to be fixed. While the extension of the notion of support is unproblematic, the definition of possibilities as maximal states supporting a formula fails, in general, due to the infinitary character of the semantics.

This leads us to reconstruct our system right from the propositional case, basing it on an inductive definition of possibilities. The resulting semantics, which we name *possibility semantics*, is an extension of the system discussed in the previous chapters, of which it retains most features, including the logic. In addition, we encounter a new logical notion, dealing with *how* a formula can be resolved, a sound and complete axiomatization of which is established.

We find that the way in which formulas provide information and raise issues in the new system remains exactly the same. The only novelty is that a formula may now propose possibilities that are strictly included in other possibilities; we hypothesize that such possibilities may be conceived of as suggestions of the shape *might p*, and we identify a third (and last) dimension of meaning in the potential of formulas to put forward such suggestions. We then define three classes of formulas (assertions, questions and conjectures) that serve only one of these three purposes, and discuss the properties of these classes.

Possibility semantics is extended in an obvious way to a first-order language, leading to very satisfactory predictions about the inquisitive behaviour of the existential quantifier; furthermore, we show that who-questions can be defined in the resulting system and that their treatment is in accordance with the partition semantics of Groenendijk's logic of interrogation.

We conclude with some remarks on first-order inquisitive logics and its connections to intuitionistic predicate logic.

Chapter 1

An introduction to inquisitive semantics

1.1 Information states and the classical update perspective

Any introduction to Inquisitive Semantics starts from the observation that, traditionally, logic has been concerned with the study of the use of language in argumentation. In order to judge the correctness of an argument moving from a set of premises to a conclusion, one has to check whether the utterances occurring in the argument are related to the previous ones in such a way as to insure the preservation of truth.

As a consequence of this approach, the *meaning* of a sentence has been identified with its truth conditions. The central notion to arbitrate the soundness of an argument is that of *entailment*: a formula φ entails a formula ψ in case the truth conditions for ψ are at least as weak as the truth conditions for φ.

However, argumentation is neither the only nor the primary use of language. Much more common is the use of language in dialogue, for the purposes of information exchange. A first, important step in the direction of the modelling of the dynamics of this use of language has been undertaken by Stalnaker (1978). There, starting from the classical truth-conditional perspective on sentences, meaning is identified with its potential to change the *common ground* of a conversation.

The *common ground* models the conversational participants' shared information as a set of possible worlds (i.e., models for the given formal language), which are conceived of as the configurations of reality that are consistent with the information which is common knowledge among the participants.

Following this interpretation, we refer to a set of models as an *information state*. Clearly, larger information states reflect a greater uncertainty about the configuration of reality, since a broader set of alternatives is contemplated; the limit case is the state consisting of *all* models, which reflect a state of complete ignorance about the issues at stake. Conversely, a narrow state reflects abundance of information, as many conceivable configurations are ruled out: the limit case is that of a singleton state, in which the information about the issues at stake is complete.

The empty state has a special status: no configuration is considered possible for reality; this reflects the fact that (the agent is Gorgias or) inconsistent information has been acquired.

Now, according to its truth conditions, a sentence splits an information state into two substates, consisting, respectively, of the models in which it is true, and those in which it is false. Now, in Stalnaker's approach, a sentence is regarded as an operator that, when applied to an information state s, returns the substate $s[\varphi]$ consisting of the models in which it is true. The state $s[\varphi]$ is often referred to as the *update* of s with φ.

If the information state in question is taken to represent the common ground of a conversation, this update process is understood as follows: after a sentence has been uttered, the fact that it is true becomes part of the participants' common knowledge; consequently, all the worlds in which the sentence is false are not consistent with the shared information anymore, and are therefore eliminated from the common ground.

The utterance of a sentence has the effect of shrinking down the common ground, making it more informed. We can thus say that, in this approach, the meaning of a sentence coincides with its *informative content*.

1.2 Limitations of the classical picture

The picture we have presented in the previous paragraph is limited in several respects.

1. First of all, only *some* of the utterances in a dialogue have the effect of providing information. Many others, such as questions, exclamations, and imperatives, do not (primarily) serve this purpose. Such sentences cannot be given a traditional truth-conditional account. Still, they make an essential contribution in dialogues. Questions, for instance, are typically used by agents to specify an issue that they would like to resolve through the interaction with the other dialogue participants; thus, questions give dialogue a direction.

2. Secondly, the classical picture does not reflect the cooperative nature of the process of updating of the common ground of a conversation: as soon as a sentence is uttered, the update takes place "automatically", regardless of the other participants' reactions. In particular, critical moves like disagreeing or doubting cannot be accounted for.

3. Another important limitation of the classical framework is that —essentially because of its inability to model questions and issues in general— it completely fails to provide proper tools to analyse how utterances of a dialogue

relate to one another and whether the dialogue is coherent. As far as the classical analysis is concerned, an arbitrary sequence of unrelated —but consistent and informative— assertions would constitute a perfectly acceptable development for a conversation.

4. Finally, in a dialogue, even informative sentences can serve other purposes as well. In particular, it is has often been observed that disjunctions and indefinites are frequently used in order to raise issues: sentences displaying this feature are called *hybrids*. We will say more on this further on.

1.3 The inquisitive programme: propositions as proposals

The aim of Inquisitive Semantics is to devise a formal system capable of modelling a broader range of meanings, such as those attached to questions and hybrid sentences, by allowing the representation of issues, together with information; such a system provides the tools for the study of dialogue as a process of raising and resolving issues, and the analysis of whether and how an utterance in a dialogue relates to the preceding ones.

Crucial in this study is the notion of *compliance*, judging whether a sentence makes a significant contribution to the process of resolution of a given issue. However central in the inquisitive programme, this notion will not be treated in this thesis. For the precise definition of compliance and its implementation in Inquisitive Semantics, the reader is referred to Groenendijk and Roelofsen (2009) and Ciardelli *et al.* (2009a).

The starting point in the construction of Inquisitive Semantics is the above observation that (at least in usual circumstances) a dialogue is a *cooperative* process that sees the participants collaborate in order to resolve one another's issues. The acknowledgement of the cooperative nature of information exchange naturally leads to regarding meanings as *proposals*: any sentence has the effect of proposing one or more *possibilities*, i.e. possible ways to update the common ground.

If the sentence proposes a single possibility, then it is an *assertion*, i.e. purely informative; still, for the proposed update to take place it is necessary that the other participants react in a positive way to the proposal - for instance, by nodding or confirming - or at least that they do not object to it, in which case the update does not take place.

If on the other hand the sentence proposes more than one possibility, then it is *inquisitive*, i.e. it invites a choice from the other participants aimed at choosing, ideally, exactly one of the proposed alternatives. If this happens, we say that the sentence is *resolved*. Of course, the other agents may not have sufficient information to pick exactly one alternative, in which case they may opt for less demanding responses, such as the union of some of the proposed alternatives.

Whatever the response of the other agents to the utterance of φ is, as soon as it is compliant with φ the common ground is ultimately updated to a subset of the union of the possibilities proposed by φ; in the worst case, if no dialogue participant is in the position to make an even partial contribution to the choice invited by φ, the common ground will be updated to the union of the possibilities proposed by φ. This shows that even an inquisitive formula has a (possibly trivial) informative component, as it proposes to eliminate from the common ground those worlds that are not included in any of the specified alternatives.

1.4 Sources of inquisitiveness: questions, disjunctions and indefinites

For an example of the inquisitive approach, consider the polar question $?p$ ("is it the case that p?"): in Inquisitive Semantics, this formula will be interpreted as inviting a choice between p and $\neg p$ aimed at updating the common ground with either assertion. Analogously - moving to the predicate setting - a *who* question like $?xP(x)$ will be interpreted as inviting a choice between alternatives, corresponding to the possible complete answers.

But questions are not the only linguistic entities that have the power to raise issues. Besides their traditional role in dealing with informative content, disjunction and the existential quantifier (the latter in the form of indefinite pronoun) are widely acknowledged to serve other purposes in natural language. It is an observation that goes back at least to Grice (1989) that a standard use of disjunction consists in the specification of alternative possibilities. This function is striking in alternative questions, like:

(1) Is Caracas in Colombia or in Venezuela?

(2) Should we call her, or should we wait?

Groenendijk (2008c), Mascarenhas (2008) and van Gool (2009) have argued that indicatives like (3) also specify alternatives if uttered with an intonation pattern that emphasizes the disjuncts.

(3) Alf or Bea will go to the party.

This is witnessed by the fact that (4) sounds a perfectly acceptable response to such an utterance.

(4) Yes, Bea will go.

Inquisitive Semantics assigns to disjunction a hybrid behaviour, that consists in *informing that at least one* of the disjuncts is the case, and *proposing* (in the basic case) *two alternatives*, corresponding to the disjuncts.

The inquisitive employment is even more common in the case of the existential quantifier. Think of sentences like the following.

(5) I put my wallet somewhere...

The purpose of an utterance of (5) is clearly not an informative one, as the informative content of the sentence is trivial. Rather, the sentence is used in order to raise the same issue expressed by question (6).

(6) Where did I put my wallet?

The same point can be made about the utterance of (7) after the bell has rung in a house.

(7) There is someone at the door!

Other people in the room are obviously already aware of the fact that someone is at the door; again, the indefinite pronoun seems to be used in order to raise the underlying *who*-issue.

This inquisitive behaviour also shows up when the existential occurs in a question. For example, in most situations the following is not interpreted as a *yes-no* question.

(8) Is anybody going to the bike trip?

Rather, a natural answer to (8) would be of the following form.

(9) Yes, Mia and I are going.

The treatment of the existential quantifier in Inquisitive Semantics is analogous to that of disjunction: an existential $\exists x\varphi(x)$ *informs* that $\varphi(d)$ is the case for at least one individual d, and *proposes* (in the basic case) the alternatives $\varphi(d)$ for d an element of the domain.

A pleasant and perhaps surprising feature of inquisitive semantics is that the treatment of questions does not require an extension of the usual logical language with additional operators: once the inquisitive behaviour of disjunction and the existential quantifier has been recognized and modelled, meanings for questions can be constructed using the expressive power of those connectives. This makes Inquisitive Semantics and its logic easy to relate and compare with other familiar semantics and logics.

1.5 Historical notes

Inquisitive semantics was first conceived by Groenendijk (2008c) and Mascarenhas (2008), who implemented a propositional system in which formulas are evaluated against pairs of valuations; we will refer to that system as the *pair semantics*. The associated logic was axiomatized by Mascarenhas (2009), while a sound and complete sequent calculus was established by Sano (2008).

Shortly afterwards, Groenendijk (2008a) and, independently, Ciardelli (2008) realized that the pair semantics could be easily generalized to a system in which formulas are evaluated against arbitrary sets of valuations, and Ciardelli (2008) argued that the pair semantics is affected by shortcomings that can be overcome in the generalized setting. It is this version of the semantics (together with a variant of it that will be required in order to deal with the first-order case) that is discussed, explored and used in this thesis.

Groenendijk (2008a) and Ciardelli (2008) remarked the existence of a tight connection between generalized inquisitive semantics and intuitionistic Kripke semantics (Kripke, 1965)[1]; this connection was exploited by Ciardelli and Roelofsen (2009) to establish a sound and complete axiomatization of the logic associated to inquisitive semantics. This was obtained by augmenting intuitionistic logic with the addition of the Kreisel-Putnam axiom scheme (Kreisel and Putnam, 1957) and the law of double negation restricted to atomic proposition letters alone. This result will be presented in section 3.2.2, where it will also be obtained independently as a particular case of a much more general and simpler theorem.

The semantics investigated in this thesis is at present being used as a framework for several linguistic applications:

1. Groenendijk and Roelofsen (2009), on inquisitive pragmatics and the notion of compliance;
2. Ciardelli *et al.* (2009b), on the modelling of *might* and free choice effects;
3. van Gool (2009), on the modelling of different intonation patterns for disjunctive questions;
4. Ciardelli *et al.* (2009a), on the computational aspects of compliance.

Another extensive linguistic work based on inquisitive semantics is being developed by Balogh (2009), who is using the pair version of inquisitive semantics to study phenomena related to focus in dialogue and question-answer relations.

[1]Mascarenhas (2008) discussed an analogous connection for the pair semantics.

Chapter 2

Propositional inquisitive semantics

2.1 Propositional inquisitive semantics and its properties

In this chapter we put the ideas exposed in the introduction into effect. We define an inquisitive semantics for a propositional language and discuss the properties of the resulting system.

We assume a language $\mathcal{L}_\mathcal{P}$, whose expressions are built up from $\bot$ and a (finite or countably infinite) set of proposition letters $\mathcal{P}$, using binary connectives $\wedge, \vee$ and $\rightarrow$. We will also make use of three abbreviations: $\neg\varphi$ for $\varphi \rightarrow \bot$, $!\varphi$ for $\neg\neg\varphi$, and $?\varphi$ for $\varphi \vee \neg\varphi$. The first is standard, while the role of the second and the third will clarified at the end of the present section.

2.1.1 Indices, States, and Support

The basic ingredients for the semantics are *indices* and *states*.

Definition 2.1.1 (Indices)**.** A $\mathcal{P}-$*index* is a subset of $\mathcal{P}$. The set of all indices, $\wp(\mathcal{P})$, will be denoted by $\mathcal{I}_\mathcal{P}$. We will simply write $\mathcal{I}$ and talk of *indices* in case $\mathcal{P}$ is clear from the context.

Definition 2.1.2 (States)**.** A $\mathcal{P}-$*state* is a set of $\mathcal{P}-$indices. The set of all states, $\wp\wp(\mathcal{P})$, will be denoted by $\mathcal{S}_\mathcal{P}$. Again, reference to $\mathcal{P}$ will be dropped whenever possible.

The meaning of a sentence will be defined in terms of the notion of *support* (just as, in a classical setting, the meaning of a sentence is usually defined in terms of truth). Support is a relation between states and formulas defined as follows.

Definition 2.1.3 (Support)**.**

1. $s \models p$ iff $\forall w \in s : p \in w$
2. $s \models \bot$ iff $s = \emptyset$
3. $s \models \varphi \wedge \psi$ iff $s \models \varphi$ and $s \models \psi$
4. $s \models \varphi \vee \psi$ iff $s \models \varphi$ or $s \models \psi$
5. $s \models \varphi \rightarrow \psi$ iff $\forall t \subseteq s$: if $t \models \varphi$ then $t \models \psi$

It is clear that according to this definition, the empty state supports any formula φ, and moreover, it is the only state for which this is the case. Thus, we also refer to $\emptyset$ as the *inconsistent* state. The following two basic facts about support can be established by a straightforward induction on the complexity of φ:

Proposition 2.1.4 (Persistence)**.** If $s \models \varphi$ then for every $t \subseteq s$: $t \models \varphi$

Proposition 2.1.5 (Singleton states behave classically)**.** For any index w and formula φ:

$$\{w\} \models \varphi \iff w \models \varphi$$

where $w \models \varphi$ means: φ is classically true under the valuation w. In particular, $\{w\} \models \varphi$ or $\{w\} \models \neg\varphi$ for any formula φ.

It follows from definition 2.1.3 that the support-conditions for $\neg\varphi$ and $!\varphi$ are as follows.

Proposition 2.1.6 (Support for negation)**.**

1. $s \models \neg\varphi$ iff $\forall w \in s : w \models \neg\varphi$
2. $s \models !\varphi$ iff $\forall w \in s : w \models \varphi$

Proof. Clearly, since ! abbreviates double negation, item 2 is a particular case of item 1. To prove item 1, first suppose $s \models \neg\varphi$. Then for any $w \in s$ we have $\{w\} \models \neg\varphi$ by persistence, and thus $w \models \neg\varphi$ by proposition 2.1.5.
Conversely, if $s \not\models \neg\varphi$, then there must be $t \subseteq s$ with $t \models \varphi$ and $t \not\models \bot$. Since $t \not\models \bot$, $t \neq \emptyset$: thus, taken $w \in t$, by persistence and the classical behaviour of singleton states we have $w \models \varphi$. Since $w \in t \subseteq s$, it is not the case that $v \vdash \neg\varphi$ for all $v \in s$. □

The following construction will often be useful when dealing with cases where the set of propositional letters is infinite.

Definition 2.1.7. Let $\mathcal{P} \subseteq \mathcal{P}'$ be two sets of propositional letters. Then for any $\mathcal{P}'-$state s, the *restriction* of s to $\mathcal{P}$ is defined as $s\restriction_{\mathcal{P}} := \{w \cap \mathcal{P} \mid w \in s\}$.

The following fact, which can be established by a straightforward induction on the complexity of φ, says that whether or not a state s supports a formula φ only depends on the 'component' of s that is concerned with the letters in φ. In other terms, this insures that, when considering the meaning of a formula φ, we might just as well restrict our attention to states for the propositional letters occurring in φ.

Proposition 2.1.8 (Restriction Invariance). Let $\mathcal{P} \subseteq \mathcal{P}'$ be two sets of propositional letters. Then for any $\mathcal{P}'-$state s and any formula φ whose propositional letters are in $\mathcal{P}$:

$$s \models \varphi \iff s\upharpoonright_{\mathcal{P}} \models \varphi$$

2.1.2 Inquisitive meanings

In terms of support, we define the *possibilities* for a formula φ and in turn the *meaning* of φ. We also define the *truth-set* of φ, which embodies the *classical meaning* of φ.

Definition 2.1.9 (Truth sets, possibilities, meanings). Let φ be a formula.

1. The *truth set* of φ, denoted by $|\varphi|$, is the set of indices where φ is classically true.

2. A *possibility* for φ is a maximal state supporting φ, that is, a state that supports φ and is not properly included in any other state supporting φ.

3. The *meaning* of φ, denoted by $[\varphi]$, is the set of possibilities for φ.

Notice that $|\varphi|$ is a state, while $[\varphi]$ is a set of states. We have seen in the introduction that in inquisitive semantics, the term *proposition* is taken literally, in the sense that a formula φ is seen as *proposing* one or more ways to enhance the common ground. Formally, these 'ways' correspond to the possibilities for φ, and therefore the meaning $[\varphi]$ represents the *proposition expressed by* φ: this is in fact the terminology adopted in the more linguistically-oriented presentations of inquisitive semantics.

It may be expected that the meaning of φ would be defined as the set of all states supporting φ. Rather, though, it is defined as the set all *maximal* states supporting φ, that is, the set of all *possibilities* for φ. This is motivated by the fact that meanings in inquisitive semantics are viewed as proposals consisting of one or more alternative possibilities. If one state is included in another, we do not regard these two states as *alternatives*. This is why we are particularly interested in *maximal* states supporting a formula. Technically, however, the two approaches are equivalent: for, the next proposition shows that the meaning of φ is sufficient to determine which states support φ and which states do not.

Proposition 2.1.10 (Support and Possibilities). For any state s and any formula φ:

$$s \models \varphi \iff s \text{ is included in a possibility for } \varphi$$

Proof. If $s \subseteq t$ and t is a possibility for φ, then by persistence $s \models \varphi$. For the converse, first consider the case in which the set $\mathcal{P}$ of propositional letters is finite. Then there are only finitely many states, and therefore if s supports φ, then obviously s must be contained in a *maximal* state supporting φ, i.e. in a possibility.

If $\mathcal{P}$ is infinite, given a $\mathcal{P}-$state $s \models \varphi$, consider its restriction $s \restriction_{\mathcal{P}_\varphi}$ to the (finite!) set $\mathcal{P}_\varphi$ of propositional letters occurring in φ. By proposition 2.1.8, $s \restriction_{\mathcal{P}_\varphi} \models \varphi$, and thus $s \restriction_{\mathcal{P}_\varphi} \subseteq t$ for some $\mathcal{P}_\varphi-$state t which is a possibility for φ. Now, consider $t^+ := \{w \in \mathcal{I}_\mathcal{P} \mid w \cap \mathcal{P}_\varphi \in t\}$. For any $w \in s$ we have $w \cap \mathcal{P}_\varphi \in (s \restriction \mathcal{P}_\varphi) \subseteq t$, so $w \in t^+$ by definition of t^+; this proves that $s \subseteq t^+$. Moreover, we claim that t^+ is a possibility for φ.
First, since $t^+ \restriction_{\mathcal{P}_\varphi} = t$ and $t \models \varphi$, it follows from proposition 2.1.8 that $t^+ \models \varphi$. Now, consider a state $u \supseteq t$ with $u \models \varphi$: then $u \restriction_{\mathcal{P}_\varphi} \supseteq t^+ \restriction_{\mathcal{P}_\varphi} = t$ and moreover, again by proposition 2.1.8, $u \restriction_{\mathcal{P}_\varphi} \models \varphi$; but then, by the maximality of t it must be $u \restriction_{\mathcal{P}_\varphi} = t$. Now, for any $w \in u$, $w \cap \mathcal{P}_\varphi \in u \restriction_{\mathcal{P}_\varphi} = t$, so $w \in t^+$ by definition of t^+: hence, $u = t^+$. This proves that t^+ is indeed a possibility for φ. □

Note that, as any formula is supported at least by the empty state, this fact implies that all formulas have at least one possibility. Thus, an inquisitive meaning is always a non-empty set of states.[1]

Now, suppose a sentence φ is uttered: assuming the dialogue develops according to the compliance rules (cf. Groenendijk, 2008b; Ciardelli *et al.*, 2009a), if the proposal expressed by φ is not rejected, then the common ground will eventually be updated to the union of a set of possibilities proposed by φ, in the worst case to $\bigcup[\varphi]$; therefore, indices not in $\bigcup[\varphi]$ will ultimately be eliminated from the common ground as a consequence of the utterance of φ.

In this effect lies the informative power of a formula φ: φ proposes to eliminate indices not in $\bigcup[\varphi]$, as its acceptance implies the removal of such indices. In other words, a formula φ provides the information that the "actual world" lies in $\bigcup[\varphi]$.

In a classical setting, the informative content of φ is captured by $|\varphi|$. Hence, the following result can be read as stating that inquisitive semantics agrees with classical semantics as far as informative content is concerned, that is, inquisitive semantics gives a classical treatment of information.

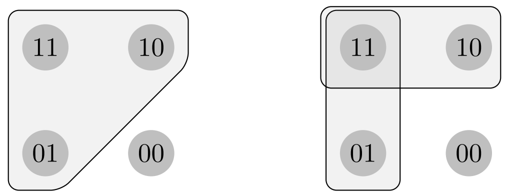
Proposition 2.1.11. For any formula φ: $\bigcup[\varphi] = |\varphi|$.

Proof. According to proposition 2.1.5, if $w \in |\varphi|$, then $\{w\} \models \varphi$. But then, by proposition 2.1.10, $\{w\}$ must be included in some $t \in [\varphi]$, whence $w \in \bigcup[\varphi]$. Conversely, any $w \in \bigcup[\varphi]$ belongs to a possibility for φ, so by persistence and the classical behaviour of singletons we must have that $w \in |\varphi|$. □

As a consequence of this result, the *informative* effect of the utterance of a formula φ can simply be rephrased as 'informing that φ is the case'.

Example 2.1.12 (Disjunction)**.** Inquisitive semantics crucially differs from classical semantics in its treatment of disjunction. This is illustrated by figures 2.1(a) and 2.1(b). These figures assume that $\mathcal{P} = \{p, q\}$; the index 11 makes both p and q true, index 10 makes p true and q false, etcetera. Figure 2.1(a) depicts the truth set—that is, the classical meaning—of $p \vee q$: the

[1] In some presentations of inquisitive semantics, such as (Groenendijk, 2009), the empty state is excluded from the semantics, and consequently meanings may be empty.



(a) $|p \vee q|$ (b) $[p \vee q]$

Figure 2.1: Classical and inquisitive meaning of $p \vee q$.

set of all indices that make either p or q, or both, true. Figure 2.1(b) depicts the proposition associated with $p \vee q$ in inquisitive semantics. It consists of two possibilities. One possibility is made up of all indices that make p true, and the other of all indices that make q true. So in the inquisitive setting, $p \vee q$ proposes two alternative ways of enhancing the common ground, and invites a response that is directed at choosing between these two alternatives.

The following inequalities, first observed by Groenendijk, provide an upper bound for the number of possibilities for a formula in terms of the number of possibilities for its immediate subformulas. If X is a set, denote its cardinality by $\#X$.

Proposition 2.1.13 (Groenendijk's inequalities)**.**

1. $\#[p] = \#[\bot] = 1$
2. $\#[\varphi \vee \psi] \leq \#[\varphi] + \#[\psi]$
3. $\#[\varphi \wedge \psi] \leq \#[\varphi]\#[\psi]$
4. $\#[\varphi \to \psi] \leq \#[\varphi]^{\#[\psi]}$

Proof. We already established the basic cases. The other inequalities follow from the following observations, whose straightforward verification is omitted.

1. Any possibility for $\varphi \vee \psi$ is a possibility for at least one of φ and ψ. This gives an injection from $[\varphi \vee \psi]$ into the union $[\varphi] \cup [\psi]$.
2. Any possibility for $\varphi \wedge \psi$ is of the form $s \cap t$ for some $s \in [\varphi]$ and some $t \in [\psi]$. This gives an injection from $[\varphi \wedge \psi]$ into the cartesian product $[\varphi] \times [\psi]$.
3. Any possibility for $\varphi \to \psi$ is of the shape Π_f for some function $f : [\varphi] \to [\psi]$, where $\Pi_f = \{w \mid \text{ for all } s \in [\varphi], \text{ if } w \in s \text{ then } w \in f(s)\}$. This gives an injection from $[\varphi \to \psi]$ into the set $[\psi]^{[\varphi]}$ of functions from $[\varphi]$ to $[\psi]$.

□

In particular, we have the following corollary, that we could also have concluded from the proof of proposition 2.1.10.

Corollary 2.1.14. Any formula has a finite number of possibilities.

2.1.3 Inquisitiveness and Informativeness

Recall once again that in inquisitive semantics, a meaning $[\varphi]$ is thought of as a proposal to change the common ground of a conversation. If $[\varphi]$ contains more than one possibility, then it invites a choice between alternative possibilities, and thus we say that φ is *inquisitive*.

On the other hand we have seen that as soon as $\bigcup[\varphi] \neq \mathcal{I}$ —that is, if the possibilities for φ do not cover the entire space— the formula φ proposes the elimination of some indices from the common ground: in this case we say that φ is *informative*.

Definition 2.1.15 (Inquisitiveness and informativeness)**.**

- φ is *inquisitive* iff $[\varphi]$ contains at least two possibilities;
- φ is *informative* iff $[\varphi]$ proposes to eliminate certain indices: $\bigcup[\varphi] \neq \mathcal{I}$

Definition 2.1.16 (Questions and assertions)**.**

- φ is a *question* iff it is not informative;
- φ is an *assertion* iff it is not inquisitive.

Definition 2.1.17 (Contradictions and tautologies)**.**

- φ is a *contradiction* iff it is only supported by the inconsistent state, i.e., $[\varphi] = \{\emptyset\}$
- φ is a *tautology* iff it is supported by all states, i.e., $[\varphi] = \{\mathcal{I}\}$

It is easy to see that a formula is a contradiction iff it is a classical contradiction. This does not hold for tautologies. Classically, a formula is tautological iff it is not informative. In the present framework, a formula is tautological iff it is neither informative nor inquisitive. Classical tautologies may well be inquisitive.

Example 2.1.18 (Questions)**.** Figure 2.2 depicts the propositions expressed by the polar question $?p$, the conditional question $p \to ?q$, and the conjoined question $?p \wedge ?q$. For instance, consider the first question: $?p = p \vee \neg p$ is an example of a classical tautology that is inquisitive: it invites a choice between two alternatives, p and $\neg p$. As such, it reflects the essential function of polar questions in natural language.

Example 2.1.19 (Disjunction, continued)**.** It is clear from figure 2.1(b) that $p \vee q$ is *both* inquisitive *and* informative: $[p \vee q]$ consists of two possibilities, which, together, do not cover the set of all indices. This means that $p \vee q$ is *neither* a question *nor* an assertion.

Groenendijk's inequalities immediately entail the following corollaries, which provide sufficient syntactic conditions for being an assertion.

Corollary 2.1.20. For any propositional letter p and formulas φ, ψ:

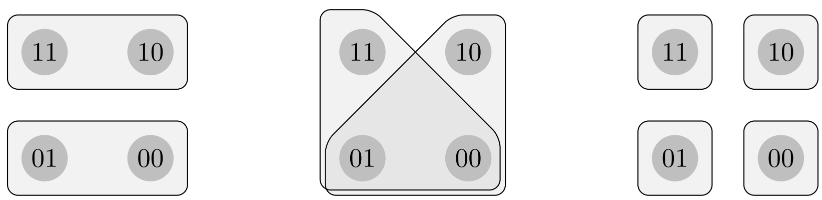


Figure 2.2: A polar question, a conditional question, and a conjoined question.

1. p is an assertion;
2. $\bot$ is an assertion;
3. if φ, ψ are assertions, then $\varphi \wedge \psi$ is an assertion;
4. if ψ is an assertion, then $\varphi \to \psi$ is an assertion.

Corollary 2.1.21. Any negation is an assertion.

Of course, a declarative $!\varphi$ is also always an assertion. Using corollary 2.1.20 inductively we obtain the following corollary showing that disjunction is the only source of inquisitiveness in our propositional language.[2]

Corollary 2.1.22. Any disjunction-free formula is an assertion.

We will now look more closely at the properties of questions and assertions and at the role of the operators ? and !, which was not duly clarified until now.

Definition 2.1.23 (Equivalence)**.**
Two formulas φ and ψ are *equivalent*, in symbols $\varphi \equiv \psi$, in case $[\varphi] = [\psi]$.

It follows immediately from proposition 2.1.10 that $\varphi \equiv \psi$ if and only if φ and ψ are supported by the same states.

Proposition 2.1.24 (Characterization of questions)**.**
For any formula φ, the following are equivalent:

1. φ is a question
2. φ is a classical tautology
3. $\neg\varphi$ is a contradiction
4. $\varphi \equiv ?\varphi$

[2]In the first-order case there will be a close similarity between disjunction and the existential quantifier, and the latter will be a source of inquisitiveness as well.

Proof. Equivalence ($1 \Leftrightarrow 2$) follows from the definition of questions and proposition 2.1.11. ($2 \Leftrightarrow 3$) and ($4 \Rightarrow 3$) are immediate from the fact that a formula is a contradiction in the inquisitive setting just in case it is a classical contradiction. For ($3 \Rightarrow 4$), note that for any state s, $s \models ?\varphi$ iff $s \models \varphi$ or $s \models \neg\varphi$. But since $\neg\varphi$ is a contradiction, the latter case is excluded (unless $s = \emptyset$, but then s supports everything) and thus $s \models ?\varphi \iff s \models \varphi$. In other words, $\varphi \equiv ?\varphi$. □

Note that an interrogative $?\varphi = \varphi \vee \neg\varphi$ is always a classical tautology, and therefore, by the equivalence ($1 \Leftrightarrow 2$), always a question. Furthermore, the equivalence ($1 \Leftrightarrow 4$) guarantees that $?\varphi \equiv ??\varphi$, which means that ? is idempotent.

Proposition 2.1.25 (Characterization of assertions)**.**
For any formula φ, the following are equivalent:

1. φ is an assertion
2. if $s_j \models \varphi$ for all $j \in J$, then $\bigcup_{j \in J} s_j \models \varphi$
3. $|\varphi| \models \varphi$
4. $\varphi \equiv !\varphi$
5. $[\varphi] = \{|\varphi|\}$

Proof. ($1 \Rightarrow 2$) Suppose φ is an assertion and let t be the unique possibility for φ. If $s_j \models \varphi$ for all $j \in J$, then by proposition 2.1.10 each s_j must be a subsets of t, whence also $\bigcup_{j \in J} s_j \subseteq t$. Thus, by persistence, $\bigcup_{j \in J} s_j \models \varphi$.

($2 \Rightarrow 3$) By proposition 2.1.5, $\{w\} \models \varphi$ iff $w \in |\varphi|$. Then if φ satisfies condition (2), $|\varphi| = \bigcup_{w \in |\varphi|} \{w\} \models \varphi$.

($3 \Rightarrow 4$) Suppose $|\varphi| \models \varphi$; by proposition 2.1.10, $|\varphi|$ must be included in some possibility s for φ; but also, by corollary 2.1.11, $s \subseteq |\varphi|$, whence $|\varphi| = s \in [\varphi]$. Moreover, since any possibility for φ must be included in $|\varphi|$ we conclude that $|\varphi|$ must be the *unique* possibility for φ. Thus, $[\varphi] = \{|\varphi|\}$.

($4 \Rightarrow 1$) Immediate, since we remarked above that a declarative $!\varphi$ is always an assertion.

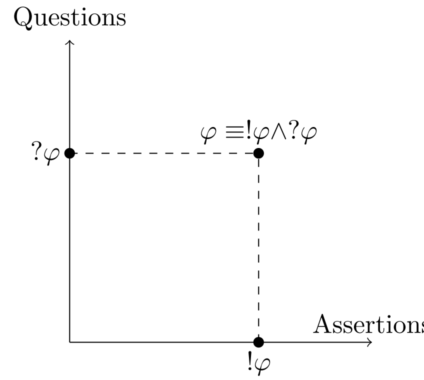
($1 \Leftrightarrow 5$) If φ is an assertion, it has only one possibility, and by the equality $\bigcup[\varphi] = |\varphi|$ (proposition 2.1.11) this unique possibility must be $|\varphi|$. The converse is trivial.

□

Note that ($1 \Leftrightarrow 5$) states that a formula is an assertion if and only if its meaning consists of its classical meaning. In this sense, assertions are precisely those formula that behave classically. Also note that ($1 \Leftrightarrow 4$), together with the fact that $!\varphi$ is always an assertion, implies that $!\varphi \equiv !!\varphi$. That is, ! is idempotent.

Moreover, let us remark that by corollary 2.1.21 and proposition 2.1.25, the assertion $!\varphi$ expresses precisely the informative content of the formula φ.

We have seen that the operators ! and ? applied to a formula produce, respectively, an assertion and a question. Proposition 2.1.25.4 shows that the application of ! to an assertion does not alter its meaning, and proposition 2.1.24 shows the same thing for the question mark operator. As a consequence, we have seen that these operators are idempotent.



In all these respects [3], the operators ! and ? act like projections on the 'planes' of assertions and questions, respectively. Moreover, the following proposition shows that the inquisitive meaning of a formula φ is completely determined by its 'purely informative component' $!\varphi$ and its 'purely inquisitive component' $?\varphi$.

Proposition 2.1.26 (Decomposition in pure components)**.** For any formula φ, $\varphi \equiv\, !\varphi \wedge ?\varphi$.

Proof. We must show that for any state s, $s \models \varphi$ iff $s \models\, !\varphi \wedge ?\varphi$. Obviously, it suffices to consider non-empty states. Suppose $s \neq \emptyset$ supports $!\varphi \wedge ?\varphi$. Then, since $s \models ?\varphi$, s must support one of φ and $\neg\varphi$; but since $s \models \neg\neg\varphi$, s cannot support $\neg\varphi$. Thus, we have that $s \models \varphi$. The converse is immediate by the definitions of ! and ? and proposition 2.1.6. □

2.2 Inquisitive semantics and intuitionistic Kripke semantics

In this section we notice the existence a tight connection between inquisitive semantics and intuitionistic Kripke semantics, a connection which will turn out to be an important tool for the study of the logic that inquisitive semantics gives rise to.

Both inquisitive and intuitionistic semantics have a natural interpretation in terms of information and information growth. For, inquisitive states and points of an intuitionistic Kripke model can both be conceived of as information states. Formulas are evaluated with respect to such information states, and support/satisfaction at a state is defined partly in terms of the information available at the state, and partly (namely, implications) in terms of possible future information states, represented by substates in inquisitive semantics and by successors in Kripke semantics.

The reader familiar with intuitionistic logic will have noticed the close similarity between clauses in the recursive definition of support and those defining

[3] And others: in section 3.1.1 we shall see that, in terms of entailment, $!\varphi$ is the assertion that "best approximates" φ.

satisfaction on intuitionistic Kripke models. This analogy has a formal counterpart: in fact, inquisitive support amounts to satisfaction on a suitable Kripke model.

Definition 2.2.1 (Kripke model for inquisitive semantics)**.** The Kripke model for inquisitive semantics is the model $M_I = \langle W_I, \supseteq, V_I \rangle$ where $W_I := \mathcal{S} - \{\emptyset\}$ is the set of all non-empty states and the valuation V_I is defined as follows: for any letter p, $V_I(p) = \{s \in W_I \mid s \models p\}$.

Observe that M_I is a Kripke model for intuitionistic logic. For, the relation $\supseteq$ is clearly a partial order. Moreover, suppose $s \supseteq t$ and $s \in V_I(p)$: this means that $s \models p$, and so by persistence $t \models p$, which amounts to $t \in V_I(p)$. So the valuation V_I is persistent. The next lemma shows that the two semantics coincide on every non-empty state.

Proposition 2.2.2 (Inquisitive support coincides with Kripke satisfaction on M_I)**.**
For every formula φ and every non-empty state s:

$$s \models \varphi \iff M_I, s \Vdash \varphi$$

Proof. Straightforward, by induction on φ. The inductive step for implication uses the fact that an implication cannot be falsified by the empty state, as the latter supports all formulas, so that restricting the semantics to non-empty states does not make a difference. □

2.3 Inquisitive semantics over an arbitrary common ground

In section 2.1.2 we saw how the notion of support gives rise to the inquisitive meaning of a formula, consisting of a set of possibilities interpreted as the possible updates that the formula proposes. However, since there we took into account *all* indices, in fact those are the updates proposed by a formula with respect to a completely *ignorant* common ground, that is, a common ground in which no possible configuration of the world has been excluded.

In practice, one wants to be able to study the effect of utterances on any possible state of the common ground: to this end, it is important to relativize the notions of inquisitive semantics to an arbitrary information state. This is what we are going to do in this section. Since this thesis focuses mostly on the logical aspects of inquisitive semantics rather than on its applications, we will keep our discussion concise and avoid dwelling too much on these issues. These have in any case been studied in some depth by Groenendijk (2009).

Definition 2.3.1 (Meaning relative to a state)**.** Let φ be a formula, and s a state.

- A *possibility* for φ in s is a maximal substate of s supporting φ;

- The *meaning* of φ in s, denoted $[\varphi]_s$, is the set of possibilities for φ in s.

It should be clear that possibilities for a formula φ are simply possibilities for φ in the ignorant state $\mathcal{I}$, and that $[\varphi] = [\varphi]_{\mathcal{I}}$. Just like in the absolute case, it is possible to verify that a substate of a state s supports a formula φ if and only if is included in a possibility for φ in s. In particular, since the empty state is a substate of any other state and supports all formulas, a relative meaning $[\varphi]_s$ is always non-empty.

In analogy with the absolute case, we call a formula *inquisitive* in a state s in case - when uttered in s - it proposes more than one possible update, and thus it invites a choice; we call a formula *informative* in s in case –when uttered in s– it proposes to eliminate indices.

Definition 2.3.2 (Informativeness and inquisitiveness relative to a state)**.**
φ is *informative* in s iff $\bigcup[\varphi]_s \neq s$;
φ is *inquisitive* in s iff $[\varphi]_s$ contains at least two possibilities.

Needless to say, the absolute notions are the particular case of the relativized ones when $s = \mathcal{I}$. Since $[\varphi]_s$ represents the effect of the utterance of φ when the common ground is in state s, if $[\varphi]_s = [\psi]_s$ then φ and ψ are equivalent when the common ground is in state s.

Definition 2.3.3 (Equivalence relative to a state)**.** We write $\varphi \equiv_s \psi$ in case $[\varphi]_s = [\psi]_s$.

Observe that if $\varphi \equiv \psi$, then φ and ψ are supported by the same states, so $\varphi \equiv_s \psi$ for any state s: thus, absolute equivalence does not only capture the notion of 'having the same behaviour on the ignorant common ground' but also that of 'having the same behaviour under any circumstance'.

We have the following alternative characterizations of relative informativeness and inquisitiveness, analogous to propositions 2.1.24 and 2.1.25; the easy proofs closely resemble those given above for the absolute notions, and will be omitted.

Proposition 2.3.4 (Alternative characterization of informativeness)**.** For any formula φ, the following are equivalent:

1. φ is not informative in s;
2. $s \subseteq |\varphi|$;
3. $\varphi \equiv_s ?\varphi$.

Proposition 2.3.5 (Alternative characterization of inquisitiveness)**.** For any formula φ, the following are equivalent:

1. φ is not inquisitive in s;
2. if $t_i \subseteq s$ and $t_i \models \varphi$ for all $i \in I$, then $\bigcup t_i \models \varphi$.

3. $|\varphi| \cap s \models \varphi$;
4. $[\varphi]_s = \{|\varphi| \cap s\}$;
5. $\varphi \equiv_s !\varphi$.

It is worth paying some attention to the following fact, that casts some light on the intuitive meaning of the support relation. For, perhaps contrarily to the expectation of the reader, support should absolutely *not* be thought of as specifying conditions under which an agent with information state s can *truthfully utter* a sentence φ, as customary in dynamics semantics (cf. Groenendijk *et al.*, 1996).

Proposition 2.3.6 (Support, inquisitiveness, and informativeness)**.**
A state s supports a formula φ iff φ is neither informative nor inquisitive in s.

Proof. Suppose $s \models \varphi$: then s itself must be the unique possibility for φ in s, that is, $[\varphi]_s = \{s\}$; so φ is neither informative nor inquisitive in s.

Conversely, suppose φ is neither informative nor inquisitive in s. Since φ is not informative in s, $\bigcup[\varphi]_s = s$; and since φ is not informative in s, $[\varphi]_s$ is a singleton, so it must be $[\varphi] = \{s\}$. Thus by definition of possibility, s must support φ. □

This proposition shows that inquisitive support is essentially a notion of *redundancy* of a formula in a state: the definition of support can be thought of as specifying the conditions under which an utterance of φ has no effect whatsoever when the common ground is in state s.

2.4 Support as 'knowing how'

We have already seen that states in inquisitive semantics can be conceived of as information states. Traditionally, an information state s is taken to support a formula φ iff it it is known in s *that* φ *is true*. As we have already remarked, this is *not* how support should be thought of in the present setting. However, there is a closely related interpretation that *is* appropriate: $s \models \varphi$ can be thought of as saying that it is known in s *how* φ *is realized*.

The idea is that a formula can be realized in different ways: for instance, $p \vee q$ can be realized by p being true or by q being true. Thus, in order to know how $p \vee q$ is realized, one must either know that p, or that q. Note that, in this sense, 'knowing how φ' means knowing *a* reason for the truth of φ, not all. The clauses in the definition of support can be read as stating exactly what it takes to know how a formula φ is realized.[4]

Under this perspective, the basic clause in the definition of support states that atoms can only be realized in *one* way: the fact that they name must obtain.

[4]This intuition can be formalized: the 'ways of being realized' of a formula are simply a set of assertions that can be defined recursively. We shall pursue this approach in chapter 6, seeing that indeed these 'ways of being realized' correspond to possibilities, so that a formula is supported if and only if one of its realizations is known in the state.

This special character of atoms explains the fact that, as we shall see, inquisitive logic is not closed under uniform substitution: in inquisitive semantics, atoms are not placeholders denoting arbitrary meanings, but really represent "atomic" meanings lacking any inquisitive complexity.

Atoms are simply names for facts (or better, states of affairs) that may or may not obtain in the world. Then, syntax provides a way to build complex entities that express and require information about such facts.

In regards to this, proposition 2.5.2, that we will see shortly, insures that each of the alternatives proposed by a formula is expressible and thus can in fact be selected by the dialogue participants through an utterance. This is certainly a desirable feature of the system that would be lost if we were to allow atoms to be inquisitive.

Returning to the definition of support, the recursive clauses can be read as follows: one never knows 'how $\bot$' unless his information state is inconsistent; one knows 'how $\varphi \vee \psi$' by either knowing 'how φ' or knowing 'how ψ'; one knows 'how $\varphi \wedge \psi$' by knowing 'how φ' and 'how ψ'.

Finally, let us consider the support-clause for implication, which becomes particularly perspicuous under this perspective. A state s supports $\varphi \to \psi$ iff every substate of s that supports φ also supports ψ. That is, we know how $\varphi \to \psi$ is realized iff in every future information state where we know how φ is realized, we also know how ψ is realized. Thus, knowing 'how φ implies ψ' requires to know not only *that* if φ is realized then so is ψ, but also *how* the way ψ is realized depends on the way φ is realized. In other words, what is required is a method for turning knowledge as to how φ is realized into knowledge as to how ψ is realized, just like in intuitionistic logic having a proof for $\varphi \to \psi$ amounts to having a method for turning a proof for φ into a proof for ψ.

The different ways in which a formula φ may be realized are mirrored by the possibilities for φ. A possibility for φ is a *maximal* state in which φ is known to be realized in a particular way. In other words, all states in which φ is known to be realized in the same way are included in the same possibility. Thus, every possibility for φ corresponds with a particular way in which φ may be realized, and vice versa.

In the light of this intuitive interpretation, the conversational effect of an utterance can be rephrased as follows. An utterance of φ provides the information *that* φ is true and it raises the issue about *how* φ is realized.

2.5 Expressive completeness and disjunctive normal form

We conclude this chapter on the semantics with some results concerning the expressive power of connectives in inquisitive semantics. We are going to show that any possible inquisitive meaning is expressible by a formula and indeed that it can be expressed by means of negation and disjunction alone; in particular, we will see that any formula can be represented as a disjunction of assertions in such a way that its possibilities coincide with the classical meaning of the disjuncts. Moreover, we shall see that all *classical* meanings —that is, consisting of one possibility only— can be expressed by means of negation and conjunction.

In order to undertake this investigation we first need to specify what a *meaning* in a given set of propositional letters is. In inquisitive semantics, the meaning of a formula is the set of *maximal* states supporting it: thus, it is a non-empty set of pairwise incomparable states. We can then take this as the abstract definition of a meaning.

Definition 2.5.1. Let $\mathcal{P}$ be a finite set of propositional letters. A *meaning* in $\mathcal{P}$ is a (non-empty) antichain of the powerset algebra $(\wp(\mathcal{I}_\mathcal{P}), \subseteq)$. We say that a meaning is *classical* in case it contains only one element.

Clearly, the meaning $[\varphi]$ of a formula φ whose propositional letters are in $\mathcal{P}$ is a meaning in $\mathcal{P}$. The following proposition shows that, conversely, every meaning in $\mathcal{P}$ is the meaning of some formula whose propositional letters are in $\mathcal{P}$.

Proposition 2.5.2 (Expressive completeness of $\vee, \neg$)**.** Any meaning Π in the propositional letters $p_1, \ldots, p_n$ can be expressed by a formula $\chi_\Pi \in \mathcal{L}_{\{p_1,\ldots,p_n\}}$ that contains only the connectives $\neg$ and $\vee$.

Proof. Let Π be a meaning in $\mathcal{P}$. Consider an element $\pi \in \Pi$: since the set of connectives $\{\neg, \vee\}$ is complete in classical logic, we can let ψ_π be a formula in $\mathcal{L}_\mathcal{P}$ that contains only disjunction and negation, such that $|\psi_\pi| = \pi$. According to proposition 2.1.6, for any state s we have $s \models\, !\psi_\pi \iff s \subseteq |\psi_\pi| = \pi$.

Now consider the formula $\chi_\Pi := \bigvee_{\pi\in\Pi} !\psi_\pi$. Note that this is a well-defined formula: for, as the set $\mathcal{P}$ of propositional letters is finite, so is $\wp(\mathcal{I}_\mathcal{P}) = \wp\wp(\mathcal{P})$, and therefore so is Π; hence, the $\bigvee_{\pi\in\Pi} !\psi_\pi$ is a *finite* disjunction.

Clearly, χ_Π contains only the connectives $\neg$ and $\vee$. Moreover, by definition of support, a state s supports χ_Π if and only if $s \models !\psi_\pi$ for some $\pi \in \Pi$, that is, according to what we saw, if and only if it is included in some element of Π.

Since Π is a meaning, no element of Π can be included in another, so each $\pi \in \Pi$ must be a *maximal* state supporting χ_Π, i.e. a possibility for χ_Π. Moreover, since we have seen that any state supporting χ_Π is included in an element of Π, elements of Π are the *only* possibilities for χ_Π. In conclusion, $[\chi_\Pi] = \Pi$. □

In particular, any meaning is expressible by a disjunction of assertions, and in fact by a disjunction of negations. This perfectly matches our intuitive understanding that meanings in inquisitive semantics are sets of *alternatives*, which are incomparable classical meanings. Classical meanings are expressed by assertions (and always expressible by negations) while disjunction is the source of alternativehood, in the sense that a disjunction applied to incomparable classical meanings yields the proposition consisting of those meanings as alternatives.

Recall that according to corollary 2.1.22, any formula containing only negations and conjunctions is an assertion, that is, expresses a classical meaning. We are now going to see that, conversely, any classical meaning can be expressed by means of negations and conjunctions alone.

Proposition 2.5.3 (Completeness of $\wedge, \neg$ for classical meanings)**.** Let $\{\pi\}$ be a classical meaning. Then there is a formula ξ_π that contains only the connectives $\neg$ and $\wedge$ and such that $[\xi_\pi] = \{\pi\}$.

Proof. Since the set of connectives $\{\neg, \wedge\}$ is complete in classical logic, we can find a formula ξ_π which contains only negations and conjunctions and such that $|\xi_\pi| = \pi$. Now since ξ_π does not contain disjunctions, by 2.1.22 it is an assertion in inquisitive semantics, and thus by 2.1.25 we have $[\xi_\pi] = \{|\xi_\pi|\} = \{\pi\}$. $\square$

This proposition shows that a formula is an assertion if and only if it is equivalent to a formula containing only negations and conjunctions. In other words, up to equivalence, the (semantic) assertive fragment of the language coincides with the (syntactic) $\{\neg, \wedge\}$-fragment.

Observe that proposition 2.5.2 comes with an associated normal form result. For, we can choose a normal form for formulas in classical logic so that the formulas ψ_π which we use in the proof of proposition 2.5.2 are uniquely determined; moreover, given a meaning Π, we can simply fix an order of its elements: $\Pi = \{\pi_1, \ldots, \pi_n\}$.

Then the formula $\chi_\Pi = !\psi_{\pi_1} \vee \cdots \vee !\psi_{\pi_n}$ expressing Π is uniquely determined: we call this formula the *normal representation* of the meaning Π. For any formula φ, the disjunctive normal form of φ is simply the normal representation of its meaning $[\varphi]$.

Knowing the disjunctive normal form of a formula is particularly useful, as it allows to read off easily (and to compute efficiently) the meaning of the formula: for, the possibilities for the formula simply correspond to the classical meanings of the disjuncts.

However, the bottom-up way we have described for producing the normal form of a formula φ *requires* the knowledge (or the computation) of the meaning of φ, and so it neutralizes the advantages of the normal form representation. This inconvenient can be avoided by getting at a disjunctive normal form through a merely syntactic, top-down procedure on the formula φ. Such an algorithm, called *disjunctive negative translation*, will be presented in the next chapter and will play a crucial role in the study of inquisitive logic.

Chapter 3

Inquisitive logic

3.1 Inquisitive Logic

3.1.1 Definitions and basic properties

In the present chapter we shall investigate the logic that inquisitive semantics gives rise to. We begin by specifying the notions of entailment and validity arising from inquisitive semantics.

Definition 3.1.1 (Entailment and validity)**.** A set of formulas Θ *entails* a formula φ in inquisitive semantics, $\Theta \models_{\mathsf{InqL}} \varphi$, if and only if any state that supports all formulas in Θ also supports φ. A formula φ is *valid* in inquisitive semantics, $\models_{\mathsf{InqL}} \varphi$, if and only if φ is supported by all states.

If no confusion arises, we will simply write $\models$ instead of $\models_{\mathsf{InqL}}$. We will also write $\psi_1, \ldots, \psi_n \models \varphi$ instead of $\{\psi_1, \ldots, \psi_n\} \models \varphi$. Note that, as expected, $\varphi \equiv \psi$ iff $\varphi \models \psi$ and $\psi \models \varphi$.

It is clear that φ entails ψ precisely in case any possibility for φ is included in a possibility for ψ. So we can think of $\varphi \models \psi$ as meaning that ψ is resolved whenever φ is resolved.

In case ψ is an assertion, this simply means that φ already provides the information carried by ψ, so inquisitive entailment boils down to classical entailment.

Proposition 3.1.2. If ψ is an assertion, $\varphi \models \psi \iff |\varphi| \subseteq |\psi|$.

Proof. Follows from proposition 2.1.25 and the definition of entailment. □

We have already seen that the exclamation mark operator turns any formula φ into an assertion $!\varphi$ expressing the informative content of φ. Moreover, in terms of entailment, the assertion $!\varphi$ can be characterized as *the most informative assertion entailed by φ.*

Proposition 3.1.3. For any formula φ and any assertion χ, $\varphi \models \chi \iff !\varphi \models \chi$.

Proof. Fix a formula φ and an assertion χ. The right-to-left implication is obvious, since it is clear from proposition 2.1.6 that $\varphi \models !\varphi$. For the converse direction, suppose $\varphi \models \chi$. Any possibility $s \in [\varphi]$ supports φ and therefore also χ, whence by proposition 2.1.10 it must be included in a possibility for χ, which must be $|\chi|$ by proposition 2.1.25 on assertions. But then also $|\varphi| = \bigcup[\varphi] \subseteq |\chi|$ whence $!\varphi \models \chi$ by proposition 3.1.2. □

Most naturally, since a question does not provide any information, it cannot entail informative formulas.

Proposition 3.1.4. If φ is a question and $\varphi \models \psi$, then ψ must be a question as well.

Proof. If φ is a question, it must be supported by each singleton state. If moreover $\varphi \models \psi$, then ψ must also be supported by each singleton state. But then, since singletons behave like indices, ψ must be a classical tautology, that is, a question. □

Definition 3.1.5 (Logic)**.** Inquisitive logic, InqL, is the set of formulas that are valid in inquisitive semantics.

Clearly, a formula is valid in inquisitive semantics if and only if it is both a classical tautology *and* an assertion. Thus, InqL coincides with classical logic as far as assertions are concerned: in particular, InqL has the same disjunction-free fragment as classical logic.

Remark 3.1.6. If φ is disjunction-free, $\varphi \in \mathsf{InqL} \iff \varphi \in \mathsf{CPL}$.

Also, note that while InqL is closed under the *modus ponens* rule, it is *not* closed under uniform substitution. For instance, $\neg\neg p \to p \in \mathsf{InqL}$ for all proposition letters, but $\neg\neg(p \vee \neg p) \to (p \vee \neg p) \notin \mathsf{InqL}$.

InqL *is*, however, closed under a special kind of substitution: for, substituting propositional letters by assertions is always an operation that preserves validity, as we shall see. This is related to the fact that, in inquisitive semantics, atoms have the special property of being assertions (that is, if satisfying the double negation law) so it is only sound to replace them with formulas that enjoy the same property.

3.1.2 Disjunction Property, Deduction Theorem, and Compactness

In this section we will discuss a few basic properties of inquisitive logic and entailment. The first observation is an immediate consequence of the fact that support is persistent.

Proposition 3.1.7. For any formula φ, $\varphi \in \mathsf{InqL} \iff \mathcal{I} \models \varphi$

In combination with the support clause for disjunction, this proposition yields the following corollary.

Corollary 3.1.8 (Disjunction Property)**.** InqL has the *disjunction property*. That is, whenever a disjunction $\varphi \vee \psi$ is in InqL, at least one of φ and ψ is in InqL as well.

Proposition 3.1.9 (Deduction theorem)**.** For any formulae $\theta_1, \ldots, \theta_n, \varphi$:

$$\theta_1, \ldots, \theta_n \models \varphi \quad \iff \quad \theta_1 \wedge \cdots \wedge \theta_n \to \varphi \in \mathsf{InqL}$$

Proof. $\theta_1, \ldots, \theta_n \models \varphi$

$\iff$ for any $s \in \mathcal{S}$, if $s \models \theta_i$ for $1 \leq i \leq n$, then $s \models \varphi$

$\iff$ for any $s \in \mathcal{S}$, if $s \models \theta_1 \wedge \cdots \wedge \theta_n$, then $s \models \varphi$

$\iff$ $\mathcal{I} \models \theta_1 \wedge \cdots \wedge \theta_n \to \varphi$

$\iff$ $\theta_1 \wedge \cdots \wedge \theta_n \to \varphi \in \mathsf{InqL}$ □

Theorem 3.1.10 (Compactness)**.** For any set of formulae Θ and any formula φ, if $\Theta \models \varphi$ then there is a *finite* set $\Theta_0 \subseteq \Theta$ such that $\Theta_0 \models \varphi$.

Proof. Since our set $\mathcal{P}$ of propositional letters is countable, so must be Θ, so we can write $\Theta = \{\theta_k \,|\, k \in \omega\}$. Now for any $k \in \omega$, let $\gamma_k = \theta_0 \wedge \cdots \wedge \theta_k$, and define $\Gamma = \{\gamma_k \,|\, k \in \omega\}$. It is clear that Γ and Θ are equivalent, in the sense that for any state s, $s \models \Gamma \iff s \models \Theta$, so we have $\Gamma \models \varphi$. Moreover, for $k \geq k'$ we have $\gamma_k \models \gamma_{k'}$. If we can show that there is a formula $\gamma_k \in \Gamma$ such that $\gamma_k \models \varphi$, then this will mean that $\{\theta_0, \ldots, \theta_k\} \models \varphi$, and since $\{\theta_0, \ldots, \theta_k\}$ is a finite subset of Θ we will be done.

For any $k \in \omega$ let $\mathcal{P}_k$ be the set of propositional letters occurring in φ or in γ_k. By the definition of the formulas γ_k, it is clear that for $k \leq k'$ we have $\mathcal{P}_k \subseteq \mathcal{P}_{k'}$. Now, towards a contradiction, suppose there is no $k \in \omega$ such that $\gamma_k \models \varphi$. Define $L_k := \{t \,|\, t \text{ is a } \mathcal{P}_k\text{-state with } t \models \gamma_k \text{ but } t \not\models \varphi\}$: our assumption amounts to saying that $L_k \neq \emptyset$ for all k. Then put $L := \{\emptyset\} \cup \biguplus_{k \in \omega} L_k$. Define a relation $\preceq$ on L by putting:

- $\emptyset \preceq t$ iff $t \in L_0$;
- $s \preceq t$ iff $s \in L_k$, $t \in L_{k+1}$ and $t\restriction_{\mathcal{P}_k} = s$.

Now, consider $t \in L_{k+1}$. This means that $t \models \gamma_{k+1}$ and $t \not\models \varphi$; as $\gamma_{k+1} \models \gamma_k$, we also have $t \models \gamma_k$. But then, since both γ_k and φ only use propositional letters from $\mathcal{P}_k$, by proposition 2.1.8 we have $t\restriction_{\mathcal{P}_k} \models \gamma_k$ and $t\restriction_{\mathcal{P}_k} \not\models \varphi$, which means that $t\restriction_{\mathcal{P}_k} \in L_k$.

From this it follows that $(L, \preceq)$ is a connected graph and thus clearly a tree with root $\emptyset$. Since L is a disjoint union of infinitely many non-empty sets, it must be infinite. On the other hand, by definition of $\preceq$, all the successors of a state $s \in L_k$ are $\mathcal{P}_{k+1}$-states, and there are only finitely many of those as $\mathcal{P}_{k+1}$ is finite. Therefore, the tree $(L, \preceq)$ is finitely branching.

By König's lemma, a tree that is infinite and finitely branching must have an infinite branch. This means that there must be a sequence $\langle t_k \,|\, k \in \omega \rangle$ of states in L such that for any k, $t_{k+1} \upharpoonright_{\mathcal{P}_k} = t_k$. This naturally defines a $\mathcal{P}$-state that is the "limit" of the sequence. Precisely, this state is:

$$t = \{w \in \wp(\mathcal{P}) \mid \text{there are } w_k \in t_k \text{ with } w_{k+1} \cap \mathcal{P}_k = w_k \text{ and } w = \bigcup_{k \in \omega} w_k\}$$

It is easy to check that for any k, $t \upharpoonright_{\mathcal{P}_k} = t_k$. Now, for any natural k, since $t \upharpoonright_{\mathcal{P}_k} = t_k \models \gamma_k$, by proposition 2.1.8 we have $t \models \gamma_k$; hence, $t \models \Gamma$. On the other hand, for the same reason, since $t \upharpoonright_{\mathcal{P}_0} = t_0 \not\models \varphi$, also $t \not\models \varphi$.
But this is contradicts the fact that $\Gamma \models \varphi$. So for some k we must have $\gamma_k \models \varphi$. □

We close this section on the basic properties of InqL with a simple remark about decidability.

Remark 3.1.11 (Decidability)**.** InqL is obviously decidable: given a formula $\varphi(p_1, \dots, p_n)$ whose propositional letters are among $p_1, \dots, p_n$, by propositions 2.1.8 and 3.1.7 we only have to test whether the $\mathcal{P}_{\{p_1,\dots,p_n\}}$-state $\mathcal{I}_{\{p_1,\dots,p_n\}}$ supports φ, and this is a finite procedure since $\mathcal{I}_{\{p_1,\dots,p_n\}}$ is finite and so has only finitely many substates that have to be checked to determine the support for implications.

3.2 Axiomatizing InqL

In section 2.2 we showed that inquisitive semantics coincides with Kripke semantics on a suitable intuitionistic Kripke model M_I that we called the *Kripke model for inquisitive semantics*, based on the set of non-empty states.

This observation suffices to show that the logic InqL contains intuitionistic propositional logic IPL. For suppose that $\varphi \notin$ InqL. Then there must be a non-empty state s such that $s \not\models \varphi$. But then we also have that $M_I, s \not\Vdash \varphi$, which means that $\varphi \notin$ IPL.

On the other hand, InqL is contained in classical propositional logic CPL, because any formula that is not a classical tautology is falsified by an index and therefore - by the classical behaviour of singleton states - by a singleton state in inquisitive semantics. So we have:

$$\mathsf{IPL} \subseteq \mathsf{InqL} \subseteq \mathsf{CPL}$$

Moreover, both inclusions are strict: for instance, $p \vee \neg p$ is in CPL but not in InqL, while $\neg\neg p \to p$ is in InqL but not in IPL.

Our task in the present section is to investigate exactly where InqL sits between IPL and CPL, exploring its relations to several intermediate logics and characterizing it either semantically, as the logic of a particular class of Kripke models, or syntactically, through an axiomatization.

In this section we will accomplish both tasks, and indeed in more than one way. In section 3.2.2 we will develop the connection between inquisitive semantics and intuitionistic Kripke semantics provided by the model M_I; we will characterize InqL as the logic of a class **nSAT** of Kripke models that share certain features of M_I, and we will show how this connection can be exploited to use the canonical model construction typical of intuitionistic logic to prove completeness of an axiomatization for InqL obtained by expanding intuitionistic logic with the Kreisel-Putnam axiom scheme and the double negation axioms restricted to atoms.

This completeness proof has the advantages of being totally self-contained and of explicitly constructing a canonical model, but on the other hand it is quite cumbersome and not totally transparent. A much speedier and neater proof will be given in section 3.2.3. Our approach there will be much more syntactic: we will disregard M_I completely and focus on the crucial fact that in InqL we can represent each formula as a disjunction of negations; an analysis of the syntactic ingredients needed to justify this translation will lead to a nicer result providing not one, but a whole range of possible axiomatizations for InqL. Surprisingly, this will even result in an exact characterization of *all* the axiomatizations of InqL (the precise meaning of this statement will become clear later on).

But first we will need to introduce some theoretical machinery designed to deal with objects like InqL, that is, non substitution-closed analogues of intermediate logics.

3.2.1 Intermediate logics and negative variants

Recall that an *intermediate logic* is defined as a consistent set of formulae that contains IPL and is closed under the rules of modus ponens and uniform substitution, where *consistent* simply means 'not containing $\bot$'.

Intermediate logics ordered by inclusion form a complete lattice whose meet operation amounts to intersection and whose join operation, also called *sum*, is defined as follows: if Λ_i, $i \in I$ is a family of intermediate logics, then $\Sigma_{i\in I}\Lambda_i$ is the logic axiomatized by $\bigcup_{i\in I}\Lambda_i$, that is, the closure of $\bigcup_{i\in I}\Lambda_i$ under the rules of modus ponens and uniform substitution. For more details, see Chagrov and Zakharyaschev (1997). The sum of *two* intermediate logic Λ and Λ' will also be denoted by $\Lambda + \Lambda'$.

In our investigation, however, we will meet several logics that, just like InqL, are *not* closed under uniform substitution. We shall refer to such logics as *weak intermediate logics*.

Definition 3.2.1. We say that a set of formulas L is a *weak logic* in case $\mathsf{IPL} \subseteq L$ and L is closed under modus ponens. If, additionally $L \subseteq \mathsf{CPL}$, then we call L a weak *intermediate* logic.

Both weak logics and weak intermediate logics ordered by inclusion form complete lattices as well, where again meet is intersection and the join (or sum) of a family is the weak logic axiomatized by the union, i.e. the closure of the union under modus ponens.

If L is a weak logic, we write $\varphi \equiv_L \psi$ in case $\varphi \leftrightarrow \psi \in L$.

Definition 3.2.2. Let K be a class of Kripke models (resp. frames). If Θ is a set of formulae and φ is a formula, we write $\Theta \models_K \varphi$ in case at any point in any model in K (resp. any model based on a frame in K), the following holds: if all formulas $\theta \in \Theta$ are forced, then so is φ.

We denote by $\mathsf{Log}(K)$ the set of formulae that are valid on each model (frame) in K, that is: $\mathsf{Log}(K) = \{\varphi \mid \models_K \varphi\}$.

It is straightforward to check that for any class K of Kripke models, $\mathsf{Log}(K)$ is a weak logic, and that for any class K of Kripke frames, $\mathsf{Log}(K)$ is an intermediate logic.

Notation. For any formula φ, we denote by φ^n the formula obtained from φ by replacing any occurrence of a propositional letter with its negation.

Definition 3.2.3 (Negative variant of a logic)**.** If Λ is an intermediate logic, we define its negative variant Λ^n as:

$$\Lambda^n = \{\varphi \,|\, \varphi^n \in \Lambda\}$$

In general, Λ^n will not be an intermediate logic: in fact, for any intermediate logic Λ we have $\neg\neg p \to p \in \Lambda^n$ for any propositional letter p; so, Λ^n will not be closed under uniform substitution unless $\Lambda^n = \mathsf{CPL}$, where CPL denotes classical logic.

It is, however, the case that Λ^n is always a weak intermediate logic containing Λ; this is the content of the following remark.

Remark 3.2.4. For any intermediate logic Λ, its negative variant Λ^n is a weak intermediate logic including Λ.

Proof. Fix an intermediate logic Λ. Since Λ is closed under uniform substitution, $\varphi \in \Lambda$ implies $\varphi^n \in \Lambda$ and so $\varphi \in \Lambda^n$. This shows $\Lambda \subseteq \Lambda^n$.

Moreover, if both φ and $\varphi \to \psi$ belong to Λ^n, then both φ^n and $(\varphi \to \psi)^n = \varphi^n \to \psi^n$ are in Λ which is closed under modus ponens; therefore, $\psi^n \in \Lambda$, which means that $\psi \in \Lambda^n$. This shows that Λ^n is closed under modus ponens.

Finally, if $\varphi \in \Lambda^n$ then $\varphi^n \in \Lambda \subseteq \mathsf{CPL}$; but then, since CPL is substitution-closed, $\varphi^{nn} \in \mathsf{CPL}$ and therefore also $\varphi \in \mathsf{CPL}$ because the double negation law holds in CPL. This shows that $\Lambda^n \subseteq \mathsf{CPL}$ and therefore that Λ^n is indeed a weak intermediate logic. □

The following immediate observation will turn out useful later on.

Remark 3.2.5. If a logic Λ has the disjunction property, then so does Λ^n.

For, if $\varphi \vee \psi \in \Lambda^n$, then $\varphi^n \vee \psi^n \in \Lambda$; thus, by the disjunction property, at least one of φ^n and ψ^n must be in Λ, which means that at least one of φ and ψ must be in Λ^n.

Definition 3.2.6 (Negative valuations)**.** Let F be an intuitionistic frame. A valuation V is called *negative* in case for any point w in F and for any proposition letter p:

$$(F, V), w \Vdash p \iff (F, V), w \Vdash \neg\neg p$$

We will call a model *negative* in case its valuation is negative. Observe that if M is a negative model, for any point w and formula φ we have $M, w \Vdash \varphi \iff M, w \Vdash \varphi^{nn}$.

Definition 3.2.7 (Negative variant of a model)**.** If $M = (W, R, V)$ is a Kripke model, we define the negative variant M^n of M to be model $M^n = (M, R, V^n)$ where

$$V^n(p) := \{w \in W \mid M, w \Vdash \neg p\}$$

that is, V^n makes a propositional letter true precisely where its negation was true in the original model.

A straightforward inductive proof yields the following result.

Proposition 3.2.8. For any model M, any point w and formula φ:

$$M, w \Vdash \varphi^n \iff M^n, w \Vdash \varphi$$

Remark 3.2.9. For any model M, its negative variant M^n is a negative model.

Proof. Take any point w of M and formula φ. According to the previous proposition and recalling that in intuitionistic logic triple negation is equivalent to single negation, we have $M^n, w \Vdash p \iff M, w \Vdash \neg p \iff M, w \Vdash \neg\neg\neg p \iff M^n, w \Vdash \neg\neg p$. $\square$

Definition 3.2.10. Let K be a class of intuitionistic Kripke frames. Then we denote by $\mathbf{n}K$ the class of negative K-models, i.e., negative Kripke models whose frame is in K.

Proposition 3.2.11. For any class K of Kripke frames, $\mathsf{Log}(\mathbf{n}K) = \mathsf{Log}(K)^n$.

Proof. If $\varphi \notin \mathsf{Log}(K)^n$, i.e. if $\varphi^n \notin \mathsf{Log}(K)$, then there must be a K-model M (i.e., a model based on a K-frame) and a point w such that $M, w \nVdash \varphi^n$. But then, by proposition 3.2.8 we have $M^n, w \nVdash \varphi$, and thus $\varphi \notin \mathsf{Log}(\mathbf{n}K)$ since M^n is a negative K-model.

Conversely, if $\varphi \notin \mathsf{Log}(\mathbf{n}K)$, let M be a negative K-model and w a point in M with $M, w \nVdash \varphi$. Then since M is negative, $M, w \nVdash \varphi^{nn}$. Therefore, by proposition 3.2.8, $M^n, w \nVdash \varphi^n$. But M^n shares the same frame of M, which is a K-frame: so $\varphi^n \notin \mathsf{Log}(K)$, that is, $\varphi \notin \mathsf{Log}(K)^n$. $\square$

The following result states that for any intermediate logic Λ, its negative variant Λ^n is axiomatized by a system having Λ and all the atomic double negation formulas $\neg\neg p \to p$ as axioms, and modus ponens as unique inference rule.

Proposition 3.2.12. If Λ is an intermediate logic, Λ^n is the smallest weak logic containing Λ and the atomic double negation axiom $\neg\neg p \to p$ for each propositional letter p.

Proof. We have already observed (see remark 3.2.1) that Λ^n is a weak logic containing Λ; moreover, for any letter p we have $\neg\neg\neg p \to \neg p \in \mathsf{IPL} \subseteq \Lambda$, so each atomic double negation formula is in Λ^n.

To see that Λ^n is *the smallest* such logic, let Λ' be another weak logic containing Λ and the atomic double negation axioms. Consider $\varphi \in \Lambda^n$: this means that $\varphi^n \in \Lambda$. But clearly, φ is derivable by modus ponens from φ^n and the atomic double negation axioms for letters in φ: hence, as Λ' contains Λ and the atomic double negation formulas and it is closed under modus ponens, $\varphi \in \Lambda'$. Thus, $\Lambda^n \subseteq \Lambda'$. □

With just a slight abuse of notation, we can thus identify Λ^n with a derivation system as follows.

Definition 3.2.13. We denote by Λ^n the following derivation system.

Axioms:

- all formulas in Λ
- $\neg\neg p \to p$ for all letters $p \in \mathcal{P}$

Rules:

- Modus ponens

If Θ is a set of formulae and φ is a formula, we will write $\Theta \models_{\Lambda^n} \varphi$ in case φ is derivable from the set of assumptions Θ in the system Λ^n.

In the following, we will think of Λ^n either as set of formulas or as a derivation system depending on which approach is more convenient.

As we saw, negative variants are not in general closed under uniform substitution; there is, however, a special kind of substitutions under which they *are* closed. This is a general version of the aforementioned (and not yet proven) fact that InqL is closed under substitution of the atoms by assertions.

Definition 3.2.14 (Negative substitutions)**.** We say that a substitution $(\ )^*$ is *negative* in a weak logic L in case $p^* \equiv_L \neg\neg(p^*)$ for all letters p.

For instance, a map that substitutes each atom by a negation is always negative. Now, the point is that in a negative variant Λ^n, atoms are essentially negations. Thus, if we want to preserve validity, it is crucial that we substitute them by formulas that enjoy the same property, namely, formulas that are equivalent to negations in Λ^n.

Proposition 3.2.15 (Closure of negative variants under negative substitutions)**.** Let Λ be an intermediate logic. If $\varphi \in \Lambda^n$ and $(\ )^*$ is a substitution which is negative in Λ^n, then $(\varphi)^* \in \Lambda^n$.

Proof. For the sake of simplicity, let us omit the brackets when dealing with multiple substitutions and write, for instance φ^{nn*} for $((\varphi^n)^n)^*$.

First observe that since $(\ )^*$ is negative in Λ^n, $\varphi^* \equiv_{\Lambda^n} \varphi^{nn*}$: for, the latter formula is obtained from the former by putting a double negation in front of each substitute p^* of a letter p.

Now, since $\varphi \in \Lambda^n$ we have $\varphi^n \in \Lambda$; then since φ^{nn*n} is a substitution instance of φ^n we have $\varphi^{nn*n} \in \Lambda$, whence $\varphi^{nn*} \in \Lambda^n$. But we saw that $\varphi^{nn*} \equiv_{\Lambda^n} \varphi^*$, so $\varphi^* \in \Lambda^n$ and we are done. □

3.2.2 Completeness by canonical model

As anticipated, in this section we will work our way up to a completeness result by exploiting the semantic connection between InqL and IPL provided by theorem 2.3.6, stating that inquisitive semantics amounts to intuitionistic Kripke semantics on the model M_I.

We first isolate some features of the Kripke model M_I embodying inquisitive semantics and characterize InqL as the logic of the class **nSAT** of Kripke models that share such features. We then propose an axiomatization and, in order to prove completeness, we build a canonical model based on the set of consistent theories with the disjunction property. Finally, we prove that this models belongs to the class **nSAT** and can therefore be used as a countermodel to inquisitive validity.

Saturated models

We start our investigation by isolating two properties of the frame F_I underlying M_I. One striking aspect of F_I is that any point in it can see an endpoint.

For, consider a point $s \in W_I$: this is a nonempty state, so there is an index $v \in s$; thus $\{v\} \subseteq s$, which means that $\{v\}$ is a successor of s in F_I. But clearly, the singleton states are precisely the endpoints of F_I.
We will call frames with this property E-SATURATED. The mnemonic is that such frames have *enough endpoints.*

Notation. If $F = (W, R)$ is a Kripke frame and s is a point in F, we denote by E_s the set of terminal successors of s, that is, $E_s = \{t \in W \mid sRt \ \&\ t \text{ is an endpoint}\}$.

Definition 3.2.16 (E-saturation)**.** A frame $F = (W, R)$ is E-SATURATED iff for any point $s \in W$, $E_s \neq \emptyset$.

The second feature of F_I that we identify is the following: if s is a point in F_I and E_* is a nonempty set of terminal successors of s, then there is always a 'mediating point' t which is a successor of s and has precisely E_* as its set of terminal successors.

To see this, note that the terminal successors of s are precisely the singletons subsets of s. Thus, for any non-empty subset $E_* \subseteq E_s$, the state $t := \bigcup E_*$ is a successor of s ($E_* \neq \emptyset$ guarantees that t is nonempty and thus a point in F_I) and clearly $E_t = E_*$.

We will call frames with this property I-SATURATED; the mnemonic is that I-SATURATED frames have *enough intermediate points*.

Definition 3.2.17 (Saturated frames)**.** We say that a Kripke frame $F = (W, R)$ is saturated in case it is both E-SATURATED and I-SATURATED. We denote by **SAT** the class of saturated frames.

Now, the fact that InqL is not closed under uniform substitution already shows that we cannot hope to characterize it as the logic of a class of Kripke frames.

Indeed, the model M_I is endowed with a special valuation function V_I, a striking feature of which is to be negative, in the sense defined in the previous section. This is precisely the feature that gives rise to the validity of the atomic double negation law $\neg\neg p \to p$ in inquisitive logic.

In order to see that V_I is negative, suppose $s \notin V_I(p)$, that is, suppose $s \not\models p$. This means that there is an index $i \in s$ with $p \notin i$; but then by the classical behaviour of singletons we have $\{i\} \models \neg p$, whence $M_I, \{v\} \Vdash \neg p$; finally, since $v \in s$, the state $\{v\}$ is a successor of s, so $M_I, s \nVdash \neg\neg p$. This shows that p and $\neg\neg p$ are forced at exactly the same points in M_I.

We are now ready to show the crucial result that paves the way to the completeness proof: InqL is precisely the logic of the class **nSAT** of negative saturated models. In fact, something stronger holds: inquisitive entailment coincides with entailment on negative saturated models.

Theorem 3.2.18 (Correspondence theorem)**.** For any set of formulae Θ and any formula φ:

$$\Theta \models_{\mathsf{InqL}} \varphi \iff \Theta \models_{\mathbf{nSAT}} \varphi$$

In the course of the present section we have seen that M_I itself is a negative saturated model. But there is more to it: the following, crucial lemma states that M_I is also the "most general" negative saturated model, in the sense that any situation arising in a negative saturated model is in fact already present in M_I.

Lemma 3.2.19. For any negative saturated model M, there is a p-morphism η from M to M_I.

Proof. Let $M = (W, R, V)$ be a finite negative saturated model. For any endpoint e of M, denote by i_e the valuation $i_e = \{p \in P \mid e \in V(p)\}$ consisting of those letters true at e. Define our candidate p-morphism η as follows:

$$\eta(w) = \{i_e \mid e \in E_w\}$$

In the first place, since M is E-SATURATED, for any $w \in W$ we have $E_w \neq \emptyset$ and so $\eta(w) \neq \emptyset$; this insures that indeed $\eta(w) \in W_I$, so that the map $\eta : W \to W_I$ is at least well-defined. It remains to check that η is a p-morphism. Fix any $w \in W$:

- **Propositional Letters**. Take any propositional letter p. If $M, w \Vdash p$, then by persistence we have $M, e \Vdash p$ for any $e \in E_w$; this implies $p \in i$ for any index $i \in \eta(w)$ and so $\eta(w) \models p$, whence $M_I, \eta(w) \Vdash p$.

  Conversely, suppose $M, w \nVdash p$. Then since the valuation V is negative, $M, w \nVdash \neg\neg p$, so there must be a successor v of w with $M, v \Vdash \neg p$. Exploiting again the fact that M is finite, take a point $e \in E_v$: by persistence it must be $M, e \Vdash \neg p$, whence $p \notin i_e$. But by the transitivity of R we also have $e \in E_w$, so $i_e \in \eta(w)$: thus $\eta(w) \not\models p$, whence $M_I, \eta(w) \nVdash p$.

- **Forth Condition**. Suppose wRv: then since our accessibility relation is transitive, $E_w \supseteq E_v$ and thus also $\eta(w) \supseteq \eta(v)$.

- **Back Condition**. Suppose $\eta(w) \supseteq t$: we must show that there is some successor v of w such that $\eta(v) = t$.

  Now, since t is a non-empty subset of $\eta(w) = \{i_e \mid e \in E_w\}$, there must be some non-empty subset $E_* \subseteq E_w$ such that $t = \{i_e \mid e \in E_*\}$.

  Then, since M is I-SATURATED there must be a successor v of w with $E_v = E_*$. We thus have: $\eta(v) = \{i_e \mid e \in E_v\} = \{i_e \mid e \in E_*\} = t$. So we have found a successor v of w with the required properties.

□

This lemma provides the key to the proof of theorem 3.2.18.

Proof of theorem 3.2.18. Suppose $\Theta \not\models_{\mathsf{InqL}} \varphi$. Then there is some state s such that $s \models \Theta$ but $s \not\models \varphi$. Now, s must be non-empty, because the empty state supports every formula. So by proposition 2.2.2 we have $M_I, s \Vdash \Theta$ but $M_I, s \nVdash \varphi$. Since M_I is a negative saturated model, this shows that $\Theta \not\models_{\mathbf{nSAT}} \varphi$.

Conversely, suppose $\Theta \not\models_{\mathbf{nSAT}} \varphi$. This means that, for some negative saturated model M and some point w in M we have $M, w \Vdash \Theta$ while $M, w \nVdash \varphi$. But according to lemma 3.2.19 there is a p-morphism $\eta : M \to M_I$, and since satisfaction is invariant under p-morphisms, we have $M_I, \eta(w) \Vdash \Theta$ and $M_I, \eta(w) \nVdash \varphi$.

Hence, again since support amounts to satisfaction on M_I, we have $\eta(w) \models \Theta$ but $\eta(w) \not\models \varphi$, which shows that $\Theta \not\models_{\mathsf{InqL}} \varphi$. The proof of theorem 3.2.18 is thus complete. □

In particular, the theorem establishes the equality $\mathsf{InqL} = \mathsf{Log}(\mathbf{nSAT})$, whence by proposition 3.2.11 we have the following corollary.

Corollary 3.2.20. $\mathsf{InqL} = \mathsf{Log}(\mathbf{SAT})^n$.

If we were willing to input some known facts about certain intermediate logics, at this point we would already be in the position to give a sound and complete axiomatization of InqL. For, the following holds.

Proposition 3.2.21. If $\Lambda = \mathsf{Log}(K)$ where K is a class of Kripke frames with $\{F_I\} \subseteq K \subseteq \mathbf{SAT}$, then $\Lambda^n = \mathsf{InqL}$.

Proof. Let Λ be as above. Since $M_I \in \mathbf{n}\{F_I\} \subseteq \mathbf{n}K \subseteq \mathbf{nSAT}$, using theorem 3.2.18 and the fact that M_I is a *negative* model on the frame F_I we have:

$$\mathsf{InqL} = \mathsf{Log}(M_I) \supseteq \mathsf{Log}(\mathbf{n}K) \supseteq \mathsf{Log}(\mathbf{nSAT}) = \mathsf{InqL}$$

So $\Lambda^n = \mathsf{Log}(K)^n = \mathsf{Log}(\mathbf{n}K) = \mathsf{InqL}$. □

Now, as the experienced reader might know, the Kreisel-Putnam logic KP (Kreisel and Putnam, 1957) axiomatized by the axiom

$$\texttt{KP} \qquad (\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$$

is known to be complete for a class **FKP** of Kripke frames called *finite Kreisel-Putnam frames* (see, for instance, Chagrov and Zakharyaschev, 1997, page 55 and theorem 5.44) which is easily seen to satisfy the hypotheses of the previous proposition, so $\mathsf{KP}^n = \mathsf{InqL}$. However, we will not go into the definition of **FKP** here: for, in the next section we will give a direct, self-contained proof of this result; on the other hand, the reader looking for a short and simple proof will find one in section 3.2.3.[1]

We can also use our completeness result to show the promised fact that substituting atoms by arbitrary assertions is sound in InqL.

Corollary 3.2.22 (Closure of InqL under substitution by assertions)**.** Let $(\ )^*$ be substitution map such that the substitute p^* of each propositional letter is an assertion. Then if $\varphi \in \mathsf{InqL}$ also $\varphi^* \in \mathsf{InqL}$.

Proof. Recall that, according to proposition 2.1.25, assertions are precisely the formulas that are equivalent to their double negation in InqL. This means that the substitution $(\ )^*$ is negative in $\mathsf{InqL} = \mathsf{Log}(\mathbf{SAT})^n$, in the sense of definition 3.2.14. Thus, the claim follows from proposition 3.2.15. □

Completeness proof by canonical model

In this chapter we will prove the equality $\mathsf{KP}^n = \mathsf{InqL}$ claimed at the end of the previous section by constructing a canonical model. We start with the matter of soundness and prove that any substitution instance of the Kreisel-Putnam axiom $\texttt{KP}$ is valid in inquisitive semantics, whence $\mathsf{KP} \in \mathsf{InqL}$.

Lemma 3.2.23. $\texttt{KP}^* \in \mathsf{InqL}$ for any substitution instance $\texttt{KP}^*$ of $\texttt{KP}$.

Proof. Consider an instance $(\neg\psi \to \chi \vee \xi) \to (\neg\psi \to \chi) \vee (\neg\psi \to \xi)$ of the scheme $\texttt{KP}$. Suppose towards a contradiction that s does not support this formula. Then there must be a substate $t \subseteq s$ such that $t \models \neg\psi \to \chi \vee \xi$ but $t \not\models \neg\psi \to \chi$ and $t \not\models \neg\psi \to \xi$.

The fact that $t \not\models \neg\psi \to \chi$ implies that there is a substate $u \subseteq t$ with $u \models \neg\psi$ but $u \not\models \chi$; similarly, since $t \not\models \neg\psi \to \xi$ there is $u' \subseteq t$ with $u' \models \neg\psi$ but $u' \not\models \xi$.

[1] Another logic to which the previous proposition apply is Medvedev logic ML, which we will discuss in detail in section 3.4.

According to corollary 2.1.21, $\neg\psi$ is an assertion, so by proposition 2.1.25 we have $u \cup u' \models \neg\psi$. But $u \cup u'$ cannot support χ, otherwise by persistency we would have $u \models \chi$, which is not the case; similarly, $u \cup u'$ cannot support ξ, and thus also $u \cup u' \not\models \chi \vee \xi$.

But since $u \subseteq t$ and $u' \subseteq t$ we have $u \cup u' \subseteq t$: this shows that $t \not\models \varphi \to \chi \vee \xi$, contrary to assumption. □

This immediately yields the soundness of the derivation system KP^n for InqL.

Proposition 3.2.24 (Soundness)**.** For any set of formulae Θ and any formula φ, if $\Theta \vdash_{\mathsf{KP}^n} \varphi$ then $\Theta \models_{\mathsf{InqL}} \varphi$.

Proof. Suppose $\Theta \vdash_{\mathsf{KP}^n} \varphi$: this means that there is a derivation of φ from formulae in Θ, axioms of intuitionistic logic, atomic double negation axioms and instances of KP, which uses modus ponens as only inference rule.

But we have already observed that $\mathsf{IPL} \subseteq \mathsf{InqL}$ and that $\neg\neg p \to p \in \mathsf{InqL}$ for $p \in \mathcal{P}$. Moreover, according to lemma 3.2.23, any substitution instance of KP is valid in inquisitive logic.

By the semantics of implication, the set of formulas supported by a state is closed under modus ponens. Therefore, fixing an arbitrary state s that supports all formulae in Θ, a straightforward induction on the length of the proof suffices to show that all formulas occurring in the proof of φ from Θ must be supported by s, in particular φ itself. □

Let us now turn to the completeness direction. We shall first prove simple completeness and only then strengthen this result to strong completeness.

Theorem 3.2.25 (Completeness theorem)**.** $\mathsf{InqL} \subseteq \mathsf{KP}^n$.

We shortly recall the elementary notions needed for the canonical model construction.

Definition 3.2.26. Let Θ be a set of formulas.

1. Θ is a *KP^n−theory* if it is closed under deduction in KP^n: for all φ, if $\Theta \vdash_{\mathsf{KP}^n} \varphi$, then $\varphi \in \Theta$.

2. Θ has the *disjunction property* if whenever a disjunction $\varphi \vee \psi$ is in Θ, at least one of φ and ψ is in Θ.

3. Θ is *(KP^n-)consistent* in case $\Theta \nvdash_{\mathsf{KP}^n} \bot$.

4. Θ is *complete* in case for any φ, exactly one of φ and $\neg\varphi$ is in Θ.

We then have the usual Lindenbaum-type lemma:

Lemma 3.2.27. If $\Theta \nvdash_{\mathsf{KP}^n} \varphi$, then there is a consistent KP^n−theory Γ with the disjunction property such that $\Theta \subseteq \Gamma$ and $\varphi \notin \Gamma$.

Proof. The proof is the usual one, but we will spell out the details for the sake of exhaustiveness. Let $(\psi_n)_{n\in\omega}$ be an enumeration of all formulae. Define:

$$\begin{array}{rcl} \Gamma_0 & = & \Theta \\ \Gamma_{n+1} & = & \begin{cases} \Gamma_n \cup \{\psi_n\} & \text{if } \Gamma_n \cup \{\psi_n\} \nvdash \varphi \\ \Gamma_n & \text{otherwise} \end{cases} \\ \Gamma & = & \bigcup_{n\in\omega} \Gamma_n \end{array}$$

Obviously, $\Theta \subseteq \Gamma$. It is immediate to check that by induction that for any n, $\Gamma_n \nvdash \varphi$, whence $\Gamma \nvdash \varphi$ and in particular $\varphi \notin \Gamma$.

Moreover, Γ is an KP^n-theory. For, suppose $\psi_n \notin \Gamma$: then $\psi_n \notin \Gamma_{n+1}$, which is only possible in case $\Gamma_n \cup \{\psi_n\} \vdash \varphi$; but then also $\Gamma \cup \{\psi_n\} \vdash \varphi$: therefore $\Gamma \nvdash \psi_n$, since otherwise it would follow that $\Gamma \vdash \varphi$, which is not the case.

Finally, Γ has the disjunction property. For, suppose ψ, ψ' are not in Γ. This implies that $\Gamma \cup \{\psi\} \vdash \varphi$ and $\Gamma \cup \{\psi'\} \vdash \varphi$; but then also $\Gamma \cup \{\psi \vee \psi'\} \vdash \varphi$: this entails $\psi \vee \psi' \notin \Gamma$, since otherwise we would have $\Gamma \vdash \varphi$, which is not the case. □

We will proceed through the construction of a canonical model for KP^n in the way that is customary for intermediate logics: the points will be the consistent KP^n-theories with the disjunction property ordered by inclusion, and the valuation will be given by membership.

Definition 3.2.28 (Canonical model for KP^n)**.** The *canonical model for* KP^n is the model $M_{\mathsf{KP}^n} = (W_{\mathsf{KP}^n}, \subseteq, V_{\mathsf{KP}^n})$, where:

- W_{KP^n} is the set of all consistent KP^n−theories with the disjunction property;
- for any propositional letter p, $V_{\mathsf{KP}^n}(p) = \{\Gamma \in W_{\mathsf{KP}^n} \mid p \in \Gamma\}$.

Note that $\subseteq$ is a partial order and that the valuation V_{KP^n} is persistent, whence M_{KP^n} is an intuitionistic Kripke model. As customary in canonical model proofs, the next step is to prove the *truth lemma*, stating that for all points in the canonical model, truth coincides with membership.

Lemma 3.2.29 (Truth Lemma)**.** For all formulas φ and points $\Gamma \in W_{\mathsf{KP}^n}$ we have $M_{\mathsf{KP}^n}, \Gamma \Vdash \varphi \iff \varphi \in \Gamma$.

Proof. By induction on φ, using lemma 3.2.27 in the inductive step for implication. □

Now, in order for M_{KP^n} to be of any use as a countermodel we have to show that it is a negative saturated model, since according to theorem 3.2.18, InqL is the logic of negative saturated Kripke models. However, in order to do so we first need to point out some properties of the canonical model.

Lemma 3.2.30 (Endpoints of M_{KP^n})**.**

1. The endpoints of M_{KP^n} are precisely the complete theories.
2. If two endpoints Δ and Δ' force the same atoms, then $\Delta = \Delta'$.
3. If the set $\mathcal{P}$ of propositional letters is finite, then for any endpoint Δ there is a formula γ_Δ such that for all endpoints Δ', $\Delta' \Vdash \gamma_\Delta \iff \Delta' = \Delta$.

Proof.

1. If Γ is a complete theory, then any proper extension of Γ must be inconsistent and thus not a point in M_{KP^n}; so Γ is an endpoint. Conversely, if Γ is consistent but not complete there must be a formula $\varphi \notin \Gamma$ such that $\Gamma \cup \{\varphi\}$ is consistent and thus, by the Lindenbaum lemma, can be extended to a point Γ' of M; so Γ is not an endpoint.
2. If two endpoints force the same atoms, then they force the same formulas. But by the truth lemma, this means that they are the same.
3. For any endpoint Δ, let $\gamma_\Delta = \mu \wedge \nu$ where μ is the conjunction of the atoms forced at Δ and ν the conjunction of the negations of the atoms not forced at Δ.

   If $\mathcal{P}$ is finite then the formula γ_Δ is well-defined. Obviously, $M_{\mathsf{KP}^n}, \Delta \Vdash \gamma_\Delta$; conversely, if an endpoint Δ' satisfies γ_Δ, then it forces *exactly* the same propositional letters as Δ, so by the previous item $\Delta = \Delta'$.

$\square$

We are now ready for the core of the completeness proof: showing that, for a finite set of propositional letters, M_{KP^n} is a negative saturated model.

Lemma 3.2.31. If the set $\mathcal{P}$ of propositional letters is finite, then $M_{\mathsf{KP}^n} \in$ **nSAT**.

Proof. Consider the canonical model M_{KP^n} built for a *finite* set of propositional letters. Let us start by showing that the valuation V_{KP^n} of the canonical model is negative. Consider any $\Gamma \in W_{\mathsf{KP}^n}$ and suppose $\Gamma' \Vdash \neg\neg p$. Then, by the truth lemma, $\neg\neg p \in \Gamma$. By the presence of the double negation axiom $\neg\neg p \to p$ in the system KP^n, this implies that $\Gamma \vdash_{\mathsf{KP}^n} p$ and so that $p \in \Gamma$, since Γ is an KP^n-theory. Hence - again by the truth lemma - $\Gamma \Vdash p$. This shows that p and $\neg\neg p$ are forced exactly at the same points in M_{KP^n}, which means that V_{KP^n} is negative.

Next, consider E-SATURATION. Take any point $\Gamma \in W_{\mathsf{KP}^n}$. It is easy to see that Γ can be extended to a complete theory Δ: in order to do so, perform the procedure described in the proof of lemma 3.2.27 with $\Theta = \Gamma$ and $\varphi = \bot$. Now, $\Gamma \subseteq \Delta$ and Δ is an endpoint by lemma 3.2.30: therefore $E_\Gamma \neq \emptyset$. This shows that M_{KP^n} is E-SATURATED.

Finally, let us come to I-SATURATION. Consider a point $\Gamma \in W_{\mathsf{KP}^n}$ and let E_* be a non-empty subset of E_Γ where, as usual, E_Γ denotes the set of endpoints of Γ. We must find a consistent KP^n-theory $\Gamma' \supseteq \Gamma$ with the disjunction property such that $E_{\Gamma'} = E_*$.

Let Γ' be the deductive closure (in KP^n) of the set $\Gamma \cup \{\neg\chi \mid \neg\chi \in \bigcap E_*\}$. We will make use of the following characterization of elements of Γ'.

Lemma 3.2.32. A formula φ is in Γ' if and only if there are $\gamma \in \Gamma$ and $\neg\chi \in \bigcap E_*$ such that $\vdash_{\mathsf{KP}^n} \gamma \wedge \neg\chi \to \varphi$.

Proof. Suppose $\varphi \in \Gamma'$: by the deduction theorem for intuitionistic logic, there are $\gamma_1, \ldots, \gamma_n \in \Gamma$, $\neg\chi_1, \ldots, \neg\chi_m \in \bigcup E_*$ such that

$$\vdash_{\mathsf{KP}^n} \gamma_1 \wedge \cdots \wedge \gamma_n \wedge \neg\chi_1 \wedge \cdots \wedge \neg\chi_m \to \varphi$$

But now, since all the γ_i are in Γ and Γ is closed under KP^n-deduction, the formula $\gamma := \gamma_1 \wedge \cdots \wedge \gamma_n$ is in Γ. Analogously, for each $\Delta \in E_*$, since all the $\neg\chi_i$ are in Δ and Δ is closed under KP^n-deduction, Δ also contains the formula $\neg\chi$ where $\chi := \chi_1 \vee \cdots \vee \chi_n$; so $\neg\chi \in \bigcap E_*$. Finally, since $\neg\chi$ is interderivable with $\neg\chi_1 \wedge \cdots \wedge \neg\chi_n$ in intuitionistic logic (and thus in KP^n) we can conclude $\vdash_{\mathsf{KP}^n} \gamma \wedge \neg\chi \to \varphi$. The converse implication is trivial. □

Proof. Proof of lemma 3.2.31, continued We are going to show that Γ' is a consistent KP^n-theory with the disjunction property (hence a point in M_{KP^n}), that it is a successor of Γ and has precisely E_* as its set of terminal successors. Thus Γ' is exactly the point whose existence we were required to show in order to establish that M_{KP^n} is I-SATURATED.

- Γ' is a KP^n-theory by definition.
- Γ' is consistent. For, suppose towards a contradiction $\bot \in \Gamma'$. Then by lemma 3.2.32 there would be $\gamma \in \Gamma$, $\neg\chi \in \bigcap E_*$ such that $\vdash_{\mathsf{KP}^n} \gamma \wedge \neg\chi \to \bot$. Since E_* is non-empty, consider a $\Delta \in E_*$: Δ is a successor of Γ (because $E_* \subseteq E_\Gamma$), so $\gamma \in \Delta$. But also $\neg\chi \in \Delta$, because $\neg\chi \in \bigcap E_*$ and $\Delta \in E_*$. Therefore, since Δ is an KP^n-theory, we would have $\bot \in \Delta$. But this is absurd, since Δ is a point of the canonical model and thus consistent by definition.
- Γ' has the disjunction property. Suppose $\varphi \vee \psi \in \Gamma'$: then there are $\gamma \in \Gamma$ and $\neg\chi \in \bigcap E_*$ such that $\vdash_{\mathsf{KP}^n} \gamma \wedge \neg\chi \to \varphi \vee \psi$. Since Γ is a KP^n-theory, $(\gamma \wedge \neg\chi \to \varphi \vee \psi) \in \Gamma$, and so since $\gamma \in \Gamma$, also $\neg\chi \to \varphi \vee \psi$ is in Γ.

  But now, since KP^n contains all instances of the Kreisler-Putnam axiom, and since Γ is closed under KP^n-deduction, $(\neg\chi \to \varphi \vee \psi) \in \Gamma$ implies $(\neg\chi \to \varphi) \vee (\neg\chi \to \psi) \in \Gamma$; thus, since Γ has the disjunction property, at least one of $\neg\chi \to \varphi$ and $\neg\chi \to \psi$ is in Γ.

  Suppose the former is the case: then $\Gamma \cup \{\neg\chi\} \vdash_{\mathsf{KP}^n} \varphi$, and since $\neg\chi \in \bigcap E_*$ this implies $\varphi \in \Gamma'$, by definition of Γ'. Instead, if it is $\neg\chi \to \psi \in \Gamma$, then reasoning analogously we come to the conclusion $\psi \in \Gamma'$.

  In either case, one of φ, ψ must be in Γ', and this proves that Γ' has the disjunction property.
- Γ' is a successor of Γ, because $\Gamma' \supseteq \Gamma$ by definition.

- $E_* \subseteq E_{\Gamma'}$. To see this, take any $\Delta \in E_*$; we are going to show that $\Gamma' \subseteq \Delta$.

  If $\varphi \in \Gamma'$, there are $\gamma \in \Gamma$ and $\neg\chi \in \bigcap E_*$ such that $\vdash_{\mathsf{KP}^n} \gamma \wedge \neg\chi \to \varphi$. Since $E_* \subseteq E_\Gamma$, we have $\Gamma \subseteq \Delta$ and thus $\gamma \in \Delta$; on the other hand, since $\Delta \in E_*$, also $\neg\chi \in \Delta$. So, both $\gamma, \neg\chi$ are in Δ which is a KP^n-theory, and therefore $\varphi \in \Delta$.

  This shows $\Gamma' \subseteq \Delta$, and thus, since Δ is an endpoint, that $\Delta \in E_{\Gamma'}$.

- $E_{\Gamma'} \subseteq E_*$. Proceed by contraposition. Consider any endpoint $\Delta \notin E_*$. Since the set $\mathcal{P}$ of propositional letters is finite, by lemma 3.2.30 there is a characteristic formula γ_Δ such that Δ is the unique endpoint satisfying γ_Δ.

  For any $\Delta' \in E_*$, $\Delta' \neq \Delta$ and thus $M_{\mathsf{KP}^n}, \Delta' \nVdash \gamma_\Delta$; since Δ' is an endpoint, this implies $M_{\mathsf{KP}^n}, \Delta' \Vdash \neg\gamma_\Delta$, whence by the truth lemma $\neg\gamma_\Delta \in \Delta'$.

  But since this is the case for any $\Delta' \in E_*$ we have $\neg\gamma_\Delta \in \bigcap E_*$ and therefore $\neg\gamma_\Delta \in \Gamma'$ by definition of Γ'.

  On the other hand, $\Delta \models \gamma_\Delta$ and thus, by the truth lemma, $\neg\gamma_\Delta \notin \Delta$. We conclude that $\neg\gamma_\Delta$ is in $\Gamma' - \Delta$, and thus $\Gamma' \not\subseteq \Delta$, which means that $\Delta \notin E_{\Gamma'}$.

□

Proof of theorem 3.2.25 (concluded). Suppose $\varphi \notin \mathsf{InqL}$. Build the canonical model M_{KP^n} for the set $\mathcal{P}_\varphi$ of propositional letters in φ: by lemma 3.2.27 there is a point $\Gamma \in W_{\mathsf{KP}^n}$ with $\varphi \notin \Gamma$. Then, the truth lemma implies $M_{\mathsf{KP}^n}, \Gamma \nVdash \varphi$.

But by the previous lemma, since $\mathcal{P}_\varphi$ is finite, $M_{\mathsf{KP}^n} \in \mathbf{nSAT}$: thus, by theorem 3.2.18 stating that $\mathsf{InqL} = \mathsf{Log}(\mathbf{nSAT})$ we have $\varphi \notin \mathsf{InqL}$. □

We can now exploit the compactness of InqL to strengthen this result to a strong completeness theorem.

Corollary 3.2.33 (Strong completeness of KP^n for InqL)**.** For any set of formulas Θ and any formula φ,

$$\Theta \models_{\mathsf{InqL}} \varphi \iff \Theta \vdash_{\mathsf{KP}^n} \varphi$$

Proof. We have already shown the soundness direction (proposition 3.2.24).

For the completeness direction, suppose $\Theta \models_{\mathsf{InqL}} \varphi$: by compactness (theorem 3.1.10) there are $\theta_1, \dots, \theta_k \in \Theta$ such that $\theta_1, \dots, \theta_k \models_{\mathsf{InqL}} \varphi$, which by the deduction theorem amounts to $\theta_1 \wedge \dots \wedge \theta_k \to \varphi \in \mathsf{InqL}$. Then by our completeness theorem $\theta_1 \wedge \dots \wedge \theta_k \to \varphi \in \mathsf{KP}^n$, whence $\Theta \vdash_{\mathsf{KP}^n} \varphi$. □

3.2.3 Completeness via disjunctive-negative translation

In this section we will define a translation DNT that turns any formula into an equivalent disjunction of negations, preserving logical equivalence with respect to inquisitive semantics. An analysis of what ingredients are needed to justify this translation will yield a whole range of different axiomatizations of InqL, one of which coincides with the one found in the previous section.

Definition 3.2.34 (Disjunctive-negative Translation).

1. $\text{DNT}(p) = \neg\neg p$
2. $\text{DNT}(\bot) = \neg\neg\bot$
3. $\text{DNT}(\psi \vee \chi) = \text{DNT}(\psi) \vee \text{DNT}(\chi)$
4. $\text{DNT}(\psi \wedge \chi) = \bigvee\{\neg(\psi_i \vee \chi_j) \,|\, 1 \leq i \leq n,\, 1 \leq j \leq m\}$

   where:

   - $\text{DNT}(\psi) = \neg\psi_1 \vee \ldots \vee \neg\psi_n$
   - $\text{DNT}(\chi) = \neg\chi_1 \vee \ldots \vee \neg\chi_m$
5. $\text{DNT}(\psi \to \chi) = \bigvee_{k_1,\ldots,k_n}\{\neg\neg\bigwedge_{1\leq i\leq n}(\chi_{k_i} \to \psi_i) \,|\, 1 \leq k_j \leq m\}$

   where:

   - $\text{DNT}(\psi) = \neg\psi_1 \vee \ldots \vee \neg\psi_n$
   - $\text{DNT}(\chi) = \neg\chi_1 \vee \ldots \vee \neg\chi_m$

Clearly, DNT always returns a disjunction of negations. Moreover, the following proposition states that DNT preserve the inquisitive meaning of formulas.

Proposition 3.2.35. For any φ, $\varphi \equiv_{\mathsf{InqL}} \text{DNT}(\varphi)$.

Proof. By induction on φ. The atomic case amounts to the validity of the atomic double negation axiom. The inductive step for disjunction is trivial, while the one for conjunction follows from the fact that IPL $\subseteq$ InqL, so that intuitionistic equivalences like the instances of the distributive laws hold in the inquisitive setting.

Finally, for the inductive step for implication we need - in addition to some intuitionistically valid equivalences - the equivalence

$$\neg\chi \to \bigvee_{1\leq i\leq k}\neg\psi_i \equiv_{\mathsf{InqL}} \bigvee_{1\leq i\leq k}(\neg\chi \to \neg\psi_i)$$

to hold for any formulas $\chi, \psi_1, \ldots, \psi_k$. Since the entailment from the right term to the left holds intuitionistically, what is needed is that any substitution instance of each of the following formulas be valid in InqL:

$$(\texttt{ND}_k) \qquad (\neg p \to \bigvee_{1\leq i\leq k}\neg q_i) \to \bigvee_{1\leq i\leq k}(\neg p \to \neg q_i)$$

But each of these formulas (except for the case $k = 1$ which is trivial) is an instance of the Kreisel-Putnam axiom KP, and therefore it is valid in inquisitive semantics according to lemma 3.2.23. □

The following proposition states that the ability to justify the "soundness" of the translation DNT is really the essential feature of InqL. Moreover, it says that InqL is the *unique* weak logic that justifies DNT and has the disjunction property.

Theorem 3.2.36. Let L be a weak intermediate logic. If $\varphi \equiv_L \text{DNT}(\varphi)$ for all formulas φ, then $\mathsf{InqL} \subseteq L$. If, additionally, L has the disjunction property, then $L = \mathsf{InqL}$.

Proof. Let L be a weak intermediate logic for which any formula φ is equivalent to $\text{DNT}(\varphi)$.

Suppose $\varphi \in \mathsf{InqL}$. Then $\text{DNT}(\varphi) \in \mathsf{InqL}$. Write $\text{DNT}(\varphi) = \neg\nu_1 \vee \cdots \vee \neg\nu_k$: since InqL has the disjunction property, we must have $\neg\nu_i \in \mathsf{InqL} \subseteq \mathsf{CPL}$ for some $1 \leq i \leq k$. But IPL coincides with CPL as far as negations are concerned (Chagrov and Zakharyaschev, 1997, corollary 2.49), so $\neg\nu_i \in \mathsf{IPL} \subseteq L$. Hence, obviously, $\text{DNT}(\varphi) \in L$, and since $\varphi \equiv_L \text{DNT}(\varphi)$, also $\varphi \in L$. This shows $\mathsf{InqL} \subseteq L$.

Moreover, suppose L also has the disjunction property. Consider a formula $\varphi \in L$: since $\varphi \equiv_L \text{DNT}(\varphi)$ we have $\text{DNT}(\varphi) \in L$. But L has the disjunction property and therefore again —using the same notation as above— $\neg\nu_i \in \Lambda^n$ for some i. Then again because all weak intermediate logics agree about negations, $\neg\nu_i \in \mathsf{InqL}$, whence $\text{DNT}(\varphi) \in \mathsf{InqL}$ and also $\varphi \in \mathsf{InqL}$. This proves $L \subseteq \mathsf{InqL}$ and therefore $L = \mathsf{InqL}$. □

This proposition shows that together, disjunctive-negative translation and disjunction property uniquely characterize InqL. This facts paves the way to a neat and more general completeness result.

In order to get there, let us go back to the proof of proposition 3.2.35. The argument used there shows that from the logical point of view, the ingredient needed to justify the translation DNT are all intuitionistically valid formulas, all instances of each axiom $\mathtt{ND}_k$, and atomic double negation axioms $\neg\neg p \to p$. Any system containing those axioms and equipped with the modus ponens rule will be able to prove the equivalence between a formula φ and its translation $\text{DNT}(\varphi)$. This suffices to prove proposition 3.2.38.

Definition 3.2.37 (ND)**.** ND is the intermediate logic axiomatized by the formulas $\mathtt{ND}_k$ for $k \in \omega$.

Proposition 3.2.38. For any logic $\Lambda \supseteq \mathsf{ND}$ and any formula φ, $\varphi \equiv_{\Lambda^n} \text{DNT}(\varphi)$.

This proposition immediately yields a whole range of intermediate logic whose negative closure coincides with inquisitive logic.

Theorem 3.2.39 (Completeness theorem)**.** $\Lambda^n = \mathsf{InqL}$ for any logic $\Lambda \supseteq \mathsf{ND}$ with the disjunction property.

Proof. Let Λ be an extension of ND with the disjunction property. Then according to the previous proposition we have $\varphi \equiv_{\Lambda^n} \text{DNT}(\varphi)$ for all φ; moreover, Λ^n has the disjunction property (see remark 3.2.5). Hence by theorem 3.2.36 we have $\Lambda^n = \mathsf{InqL}$. □

Observing that $\mathsf{ND} \subseteq \mathsf{KP}$ and recalling that both ND and KP have the disjunction property (cf. Maksimova, 1986) we immediately get two concrete instances of axiomatizations of InqL, one of which coincides with the one we found in the previous section.

Corollary 3.2.40. $\mathsf{ND}^n = \mathsf{KP}^n = \mathsf{InqL}$.

Theorem 3.2.25 gives a sufficient condition on an intermediate logic Λ for the equality $\Lambda^n = \mathsf{InqL}$ to hold. Is this condition also necessary? The answer, as we shall see in chapter 5, is *no*.

However, in section 3.4.2 we shall return to this issue and we *will* be able to give an interesting characterization of the intermediate logics whose negative variant is InqL: for, we will see that the equality $\Lambda^n = \mathsf{InqL}$ holds for an intermediate logic Λ if and only if $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$, where ML denotes Medvedev's logic of finite problems, which will be introduced in section 3.4.1.

3.3 InqL as the disjunctive-negative fragment of IPL.

The meanings of inquisitive semantics are sets of alternatives, where alternatives are incomparable classical meanings. This essential feature of the semantics is mirrored on the syntactic, logical level by the fact that any formula φ is equivalent to a disjunction of negations $\textsc{dnt}(\varphi)$.

In a way, the main result of the previous section shows that the ability to justify DNT constitutes the essence of the logic InqL. But there is even more to say: in this section we will prove that DNT is in fact a *translation* of InqL into IPL.

We will then show that the disjunctive-negative fragment of InqL coincides with the one of IPL, and that InqL is in fact isomorphic to the disjunctive-negative fragment of IPL through the translation DNT (just as CPL is isomorphic to the negative fragment of IPL through the translation mapping φ to $\neg\neg\varphi$).

Definition 3.3.1 (Translations between logics)**.** Let L, L' be two logics arising from entailment relations $\models_L$ and $\models_{L'}$ respectively. We say that a mapping t from formulas in the language of L to formulas in the language of L' is a *translation* from L to L' in case for any set of formulas Θ and any formula φ we have:

$$\Theta \models_L \varphi \iff t[\Theta] \models_{L'} t(\varphi)$$

where $t[\Theta] = \{t(\theta) \mid \theta \in \Theta\}$.

Let us call a formula *disjunctive-negative* in case it is a disjunction of negations. The following proposition says that inquisitive entailment and intuitionistic entailment agree as far as disjunctive-negative formulas are concerned.

Proposition 3.3.2. If φ is a disjunctive-negative formula and Θ a set of disjunctive-negative formulas, $\Theta \models_{\mathsf{InqL}} \varphi \iff \Theta \models_{\mathsf{IPL}} \varphi$.

Proof. Consider an arbitrary set Θ of disjunctive-negative formulas and a disjunctive-negative formula $\varphi = \neg\xi_1 \vee \cdots \vee \neg\xi_k$. If $\Theta \models_{\mathsf{InqL}} \varphi$, then by compactness and the deduction theorem there must be $\theta_1, \ldots, \theta_n \in \Theta$ such that $\theta_1 \wedge \cdots \wedge \theta_n \rightarrow \varphi \in \mathsf{InqL}$.

Now since each θ_k is a disjunction of negations and since the distributive laws hold in intuitionistic logic, in IPL we can turn $\theta_1 \wedge \cdots \wedge \theta_n$ into a disjunction of conjunctions of negations. In turn, a conjunction of negations is equivalent to a negation in intuitionistic logic. So we can find formulas $\chi_1, \ldots, \chi_m$ such that $\theta_1 \wedge \cdots \wedge \theta_n \equiv_{\mathsf{IPL}} \neg\chi_1 \vee \cdots \vee \neg\chi_m$. Hence,

$$(\theta_1 \wedge \cdots \wedge \theta_n \rightarrow \varphi) \equiv_{\mathsf{IPL}} (\neg\chi_1 \vee \cdots \vee \neg\chi_m \rightarrow \varphi) \equiv_{\mathsf{IPL}} \bigwedge_{1 \leq i \leq m} (\neg\chi_i \rightarrow \varphi)$$

Equivalence in IPL implies equivalence in InqL, so $\theta_1 \wedge \cdots \wedge \theta_n \rightarrow \varphi \in \mathsf{InqL}$ implies that $\bigwedge_{1 \leq i \leq m} (\neg\chi_i \rightarrow \varphi) \in \mathsf{InqL}$, which in turn means that for each $1 \leq i \leq n$ we have $\neg\chi_i \rightarrow \varphi \in \mathsf{InqL}$.
Writing out φ, this amounts to $\neg\chi_i \rightarrow \neg\xi_1 \vee \cdots \vee \neg\xi_k \in \mathsf{InqL}$. But since InqL contains the Kreisel-Putnam axiom, it follows that $\bigvee_{1 \leq j \leq k} (\neg\chi_i \rightarrow \neg\xi_j) \in \mathsf{InqL}$, and therefore, as InqL has the disjunction property, for some $1 \leq j \leq k$ we must have that $\neg\chi_i \rightarrow \neg\xi_j \in \mathsf{InqL} \subseteq \mathsf{CPL}$.

Now, $\neg\chi_i \rightarrow \neg\xi_j \equiv_{\mathsf{IPL}} \neg\neg(\neg\chi_i \rightarrow \neg\xi_j)$, and since CPL and IPL agree about negations (Chagrov and Zakharyaschev, 1997, corollary 2.49), also $\neg\chi_i \rightarrow \neg\xi_j \in \mathsf{IPL}$, whence *a fortiori* $\neg\chi_i \rightarrow \varphi \in \mathsf{IPL}$.

But since this can be concluded for each i, we have $\bigwedge_{1 \leq i \leq m} (\neg\chi_i \rightarrow \varphi) \in \mathsf{IPL}$, and therefore also the equivalent formula $\theta_1 \wedge \cdots \wedge \theta_n \rightarrow \varphi$ must be in IPL. But then obviously $\Theta \models_{\mathsf{IPL}} \varphi$.
The converse implication is trivial, as InqL extends IPL. ☐

As a particular case of this proposition, let us remark that for any disjunctive-negative formula φ we have $\varphi \in \mathsf{InqL} \iff \varphi \in \mathsf{IPL}$.

Corollary 3.3.3. DNT is a translation of InqL into IPL.

Proof. We have to show that for any Θ and any φ:

$$\Theta \models_{\mathsf{InqL}} \varphi \iff \mathrm{DNT}[\Theta] \models_{\mathsf{IPL}} \mathrm{DNT}(\varphi)$$

where $\mathrm{DNT}[\Theta] = \{\mathrm{DNT}(\theta) \mid \theta \in \Theta\}$. It follows from proposition 3.2.35 that $\Theta \models_{\mathsf{InqL}} \varphi \iff \mathrm{DNT}[\Theta] \models_{\mathsf{InqL}} \mathrm{DNT}(\varphi)$. But $\mathrm{DNT}(\psi)$ is always a disjunctive-negative formula. So, by proposition 3.3.2, $\mathrm{DNT}[\Theta] \models_{\mathsf{InqL}} \mathrm{DNT}(\varphi) \iff \mathrm{DNT}[\Theta] \models_{\mathsf{IPL}} \mathrm{DNT}(\varphi)$ and we are done. ☐

Observe that if the map t is a translation from a logic L to another logic L', then t naturally lifts to an embedding $\bar{t} : \mathcal{L}/_{\equiv_L} \rightarrow \mathcal{L}/_{\equiv_{L'}}$ of the Lindenbaum-Tarski algebra of L into the Lindenbaum-Tarski algebra of L', given by $\bar{t}([\psi]_{\equiv_L}) := [t(\psi)]_{\equiv_{L'}}$.[2]

[2]For more details on the issues of translations between logics we refer to (Epstein *et al.*, 1995, chapter 10)

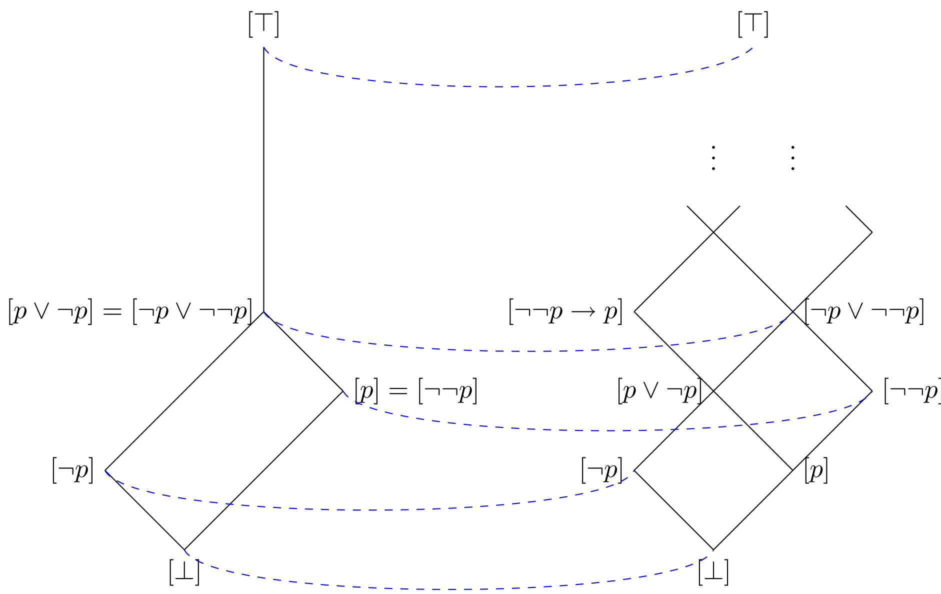


Figure 3.1: Embedding of the Lindenbaum-Tarski algebra of InqL (on the left), into the Lindenbaum-Tarski algebra of IPL (the Rieger-Nishimura lattice, on the right), for the singleton set of proposition letters $\mathcal{P} = \{p\}$.

Since we have seen that DNT is a translation from InqL to IPL, the map $\overline{\text{DNT}}$ defined by $\overline{\text{DNT}}([\psi]_{\equiv_{\mathsf{InqL}}}) = [\text{DNT}(\psi)]_{\equiv_{\mathsf{IPL}}}$ is an embedding of the Lindenbaum-Tarski algebra of InqL into the one of IPL. For the singleton set of propositional letters, this embedding is depicted in figure 3.1.

Now, for any ψ, DNT(ψ) is a disjunctive-negative formula. Conversely, consider a disjunctive-negative formula ψ. Since $\psi \equiv_{\mathsf{InqL}} \text{DNT}(\psi)$ but both ψ and DNT(ψ) are disjunctive-negative, it follows from proposition 3.3.2 that $\psi \equiv_{\mathsf{IPL}} \text{DNT}(\psi)$; in other words, we have $[\psi]_{\equiv_{\mathsf{IPL}}} = [\text{DNT}(\psi)]_{\equiv_{\mathsf{IPL}}} = \overline{\text{DNT}}([\psi]_{\equiv_{\mathsf{IPL}}})$, so $[\psi]_{\equiv_{\mathsf{IPL}}}$ is in the image of the embedding DNT.

This shows that the image of the embedding DNT is precisely the set of equivalence classes of disjunctive-negative formulas. In other words, just like CPL is isomorphic to the negative fragment of IPL, for InqL we have the following result.

Proposition 3.3.4. InqL is isomorphic to the disjunctive-negative fragment of IPL.

As a corollary of the well-known fact that CPL is isomorphic to the negative fragment of IPL we know that, for any n, there are exactly 2^{n+1} intuitionistically non-equivalent negative formulas in $\mathcal{L}_{\{p_1,\dots,p_n\}}$, just as many as there are classically non-equivalent formulas in $\mathcal{L}_{\{p_1,\dots,p_n\}}$.

Analogously, our result that InqL is isomorphic to the disjunctive-negative fragment of IPL comes with the corollary that there are exactly as many intuitionistically non-equivalent disjunctive-negative formulas in $\mathcal{L}_{\{p_1,\dots,p_n\}}$ as there are inquisitively non-equivalent formulas in $\mathcal{L}_{\{p_1,\dots,p_n\}}$.

It follows from the expressive completeness of the connectives (proposition 2.5.2) that the number of inquisitively non-equivalent formulas in $\mathcal{L}_{\{p_1,\dots,p_n\}}$ coincides with the number of distinct inquisitive meanings built up from indices in $\mathcal{I}_{\{p_1,\dots,p_n\}}$, which by definition are precisely the antichains of the powerset algebra $\wp(\mathcal{I}_{\{p_1,\dots,p_n\}})$. This algebra is isomorphic to $\wp(2^n)$, since $\mathcal{I}_{\{p_1,\dots,p_n\}} = \wp(\{p_1,\dots,p_n\})$ contains 2^n elements. Therefore, letting $D(n)$ denote the number of antichains of the powerset algebra $\wp(n)$, we have the following fact.

Corollary 3.3.5. For any n, there are exactly $D(2^n)$ intuitionistically non-equivalent disjunctive-negative formulas in $\mathcal{L}_{\{p_1,\dots,p_n\}}$.

The numbers $D(n)$ are known as Dedekind numbers (Dedekind, 1897), and although no simple formula is known for their calculation, their values for small n have been computed and are available online, see for instance: `www.research.att.com/~njas/sequences/A014466`.

The number of inquisitive meanings in one propositional letter is 5, as displayed by the above picture; with two letters we have 167 meanings, and with three the number leaps to 56130437228687557907787.

3.4 Schematic Fragment of Inquisitive Logic

We have already remarked that inquisitive logic is not closed under uniform substitution; it is natural to ask, then, what the schematic fragment of InqL is, what is the set of formulas that are schematically valid in inquisitive logic. In this section we will address this issue and we will find out that this fragment coincides with the well-known Medvedev logic of finite problems.

3.4.1 Medvedev logic

Definition 3.4.1 (Medvedev frames)**.** A Medvedev frame consists of all the non-empty subsets of some finite set X, ordered by the superset relation. In other words, a Medvedev frame is a frame of the shape $(\wp(X) - \{\emptyset\}, \supseteq)$, where X is some finite set. The class of Medvedev frames will be denoted by **Med**.

The frame F_I underlying the Kripke model for inquisitive semantics is $(\wp(\wp(\mathcal{P})) - \{\emptyset\}, \supseteq)$: so, F_I is a Medvedev frame whenever the set $\mathcal{P}$ of proposition letters is finite.

Remark 3.4.2 (Medvedev frames are saturated)**.** $\mathbf{Med} \subseteq \mathbf{SAT}$.

Proof. Medvedev frames are finite, so they are obviously E-saturated. For I-saturation, let Y be a point of a Medvedev frame, and let $E \neq \emptyset$ be a set of endpoints of Y. Then any element of E is a subset of Y, and so also $\bigcup E \subseteq Y$, that is, $\bigcup E$ is a successor of Y. Moreover, it is clear that in Medvedev frames, the endpoints are precisely the singleton sets; using this fact it is straightforward to check that the set of endpoints of $\bigcup E$ coincides precisely with E, and thus that $\bigcup E$ is the point whose existence is required by the I-saturation condition. □

Definition 3.4.3 (Medvedev Logic)**.** Medvedev logic is the logic of the class **Med** of Medvedev frames: $\mathsf{ML} := \mathsf{Log}(\mathbf{Med})$.

3.4.2 Sch(**InqL**)=**ML**

Definition 3.4.4 (Schematic fragment of a weak intermediate logic)**.** If L is a weak intermediate logic, let Sch(L) the set of formulas that are schematically valid in L, i.e., those formulas φ such that $\varphi^* \in L$ for all substitution instances φ^* of φ.

It is immediate to see that for any weak intermediate logic L, the set Sch(L) is the greatest intermediate logic included in L.

The following theorem establishes the main result of this section, namely that the schematic fragment of InqL coincides with Medvedev logic. There is, however, a subtlety that should be remarked: whereas so far we assumed that the set of atomic proposition letters $\mathcal{P}$ may be finite or countably infinite, it is at this stage important to stipulate that $\mathcal{P}$ is in fact countably infinite.

Theorem 3.4.5. Sch(InqL)=ML.

Proof. Suppose $\varphi \notin$ Sch(InqL): then there is a substitution instance φ^* of φ such that $\varphi^* \notin \mathsf{InqL}$. But then it follows from proposition 2.1.8 that φ^* can be falsified in a point of the model M_I for inquisitive semantics relative to the finite set of propositional letters $\mathcal{P}_{\varphi^*}$; and since this model is a Medvedev model, $\varphi^* \notin \mathsf{ML}$. But then, as ML is closed under uniform substitution, also $\varphi \notin \mathsf{ML}$. This shows that $\mathsf{ML} \subseteq$ Sch(InqL).

For the converse inclusion, suppose $\varphi(p_1, \dots, p_n) \notin \mathsf{ML}$. This means that there is a model $M = (F, V)$, where F is a Medvedev frame, and a point w in this model, such that $M, w \not\Vdash \varphi$. Recall that by theorem 3.2.18, $\mathsf{InqL} = \mathsf{Log}(\mathbf{nSAT})$, so in order to falsify something in InqL we just need a negative saturated countermodel.

Now, by our earlier remark, M is a saturated model. There is, however, no reason why the valuation V should be negative. Therefore, what we want to do is replace V by a negative valuation $\widehat{V}$, and pay the price for this by having to simulate the behaviour of the propositional letters $p_1, \dots, p_n$ with complex formulas $\psi_1, \dots, \psi_n$.

In order to do this, associate any point u in M with a distinct propositional letter q_u: this can be done since M is finite and we are assuming an infinite set of propositional letters. Define a new valuation $\widehat{V}$ as follows: for any point v, $v \in \widehat{V}(q_u) \iff v \subseteq u$. Then put $\widehat{M} = (F, \widehat{V})$.

Notice that the valuation $\widehat{V}$ is indeed negative. For, take any letter q_u and suppose that a certain point v is not in $\widehat{V}(q_u)$: then $v \not\subseteq u$, so we can take an element $x \in v - u$. Since $\{x\} \not\subseteq u$, $\{x\} \notin \widehat{V}(q_u)$, and since singletons are endpoints and thus behave classically we have $\widehat{M}, \{x\} \Vdash \neg q_u$; finally, since $\{x\} \subseteq v$, $\{x\}$ is a successor of v, and therefore $\widehat{M}, v \nVdash \neg\neg q_u$. So indeed $\widehat{M} \in$ **nSAT**.

We now turn to the second task, namely, find a complex formula ψ_i that simulates in $\widehat{M}$ the behaviour of the atom p_i in M. For $1 \leq i \leq n$, define $\psi_i := \bigvee_{v \in V(p_i)} q_v$. We are going to show that for any point u:

$$M, u \Vdash p_i \iff \widehat{M}, u \Vdash \psi_i$$

If $M, u \Vdash p_i$, i.e. if $u \in V(p_i)$, then since $\widehat{M}, u \Vdash q_u$ we immediately have that $\widehat{M}, u \Vdash \bigvee_{v \in V(p_i)} q_v$. That is, $\widehat{M}, u \Vdash \psi_i$.
Conversely, if $\widehat{M}, u \Vdash \psi_i$, then there is a point $v \in V(p_i)$ such that $u \in \widehat{V}(q_v)$, which in turn, by definition of $\widehat{V}$, means that $u \subseteq v$. But then, by persistence, $u \in V(p_i)$, that is, $M, u \Vdash p_i$. This proves the above equivalence. Now, it follows immediately that for any point u:

$$M, u \Vdash \varphi(p_1, \ldots, p_n) \iff \widehat{M}, u \Vdash \varphi(\psi_1, \ldots, \psi_n)$$

In particular, $\widehat{M}, w \nVdash \varphi(\psi_1, \ldots, \psi_n)$, whence $\varphi(\psi_1, \ldots, \psi_n) \notin \mathsf{Log}(\mathbf{nSAT}) =$ InqL. □

Observe that the given proof in fact establishes something stronger than the equality Sch(InqL) = ML. It shows that in order to falsify a formula $\varphi \notin$ Sch(InqL) we do not have to look at arbitrary substitution instances of φ; it suffices to take into consideration substitutions of atomic proposition letters with arbitrarily large disjunctions of atoms. In other words, we have proved the following interesting corollary.

Corollary 3.4.6. For any formula $\varphi(p_1, \ldots, p_n)$, the following are equivalent:

1. $\varphi(p_1, \ldots, p_n) \in \mathsf{ML}$;
2. $\varphi(\bigvee_{1 \leq i \leq k} p_1^i, \ldots, \bigvee_{1 \leq i \leq k} p_n^i) \in \mathsf{InqL}$ for all $k \in \omega$.
3. $\varphi(\bigvee_{1 \leq i \leq k} \neg p_1^i, \ldots, \bigvee_{1 \leq i \leq k} \neg p_n^i) \in \Lambda$ for all $k \in \omega$, where Λ is an intermediate logic with $\Lambda^n = \mathsf{InqL}$, such as ND, KP or, as we shall see, ML itself.

3.4.3 Characterization of the intermediate logics whose negative variant is InqL

In section we will strengthen the completeness result obtained in section 3.2.3 by giving a simple characterization of the logics whose negative variant is InqL. We start by noticing that Medvedev logic contains the Kreisel-Putnam logic.

Proposition 3.4.7. $\mathsf{KP} \subseteq \mathsf{ML}$.

Proof. This is the case because $\mathsf{ML} = \mathsf{Log}(\mathbf{Med})$ and the Kreisel-Putnam axiom KP is valid on any Medvedev frame. The latter fact is proven by the same argument we used to show lemma 3.2.23 stating that KP is valid in inquisitive logic. In fact, one can easily checked that the only fact about inquisitive semantics that was used in that proof is the underlying powerset structure of the model M_I, or in other words the fact that the underlying frame F_I is a Medvedev frame. □

At this point, we have at least three ways of proving the following fact.

Proposition 3.4.8. $\mathsf{ML}^n = \mathsf{InqL}$.

Proof. First, we can notice that $\{F_I\} \subseteq \mathbf{Med} \subseteq \mathbf{SAT}$ and invoke proposition 3.2.21.[3]

Second, we can combine the previous proposition informing that $\mathsf{ML} \supseteq \mathsf{KP} \supseteq \mathsf{ND}$ with the fact that ML is known to have the disjunction property (Maksimova, 1986) and then invoke theorem 3.2.39.

Third, we can proceed directly. Since $\mathsf{ND} \subseteq \mathsf{ML}$, according to proposition 3.2.38, for any φ we have $\varphi \equiv_{\mathsf{ML}^n} \textsc{dnt}(\varphi)$. So by theorem 3.2.39, $\mathsf{InqL} \subseteq \mathsf{ML}^n$.

On the other hand, we just saw that $\mathsf{ML} = \mathrm{Sch}(\mathsf{InqL}) \subseteq \mathsf{InqL}$, so InqL is a weak logic containing ML and each atomic double negation axiom, whence by proposition 3.2.12 we have $\mathsf{ML}^n \subseteq \mathsf{InqL}$. Hence $\mathsf{ML}^n = \mathsf{InqL}$. □

If we are interested in "concrete" derivation systems for InqL, the equality $\mathsf{ML}^n = \mathsf{InqL}$ is not of much use. For, as we shall discuss later on, no recursive axiomatization for ML is known. Therefore, differently from KP^n and ND^n, the logic ML^n cannot really be implemented as a derivation procedure. Nonetheless, we will see in a moment that the equality $\mathsf{ML}^n = \mathsf{InqL}$ plays an important role.

Consider a logic $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$: since the operation of taking the negative variant of a logic is clearly monotone, we have $\mathsf{InqL} = \mathsf{ND}^n = \Lambda^n = \mathsf{ML}^n = \mathsf{InqL}$ and so $\Lambda^n = \mathsf{InqL}$. The surprising fact is that the condition $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$ is not only sufficient, but also *necessary* in order to have $\Lambda^n = \mathsf{InqL}$.

Theorem 3.4.9 (Characterization of the logics whose negative variant is InqL)**.** For any intermediate logic Λ,

$$\Lambda^n = \mathsf{InqL} \iff \mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$$

Proof of theorem 3.4.9 concluded. We have already seen the right-to-left direction. For the converse implication, the next two lemmata together show that $\Lambda^n = \mathsf{InqL}$ implies $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$. □

Lemma 3.4.10. For any intermediate logic Λ, if $\Lambda^n = \mathsf{InqL}$, then $\mathsf{ND} \subseteq \Lambda$.

[3] To be precise, here we need F_I to be finite; however, given a formula φ we can always restrict to the finite set of propositional letters $\mathcal{P}_\varphi$ so that this is the case.

Proof. By contraposition, suppose $\mathsf{ND} \not\subseteq \Lambda$. Then there is a number k for which the formula $\mathtt{ND}_k := \neg p \to \bigvee_{1 \le i \le k} \neg q_i$ is not in Λ. Note that this formula is nothing but φ^n where φ denotes the formula $p \to \bigvee_{1 \le i \le k} q_i$.

But then φ^{nn} cannot be in Λ, since if it were - being Λ closed under uniform substitution - also φ^{nnn} should be in Λ, and so should be the equivalent formula φ^n. But φ^n is *not* in Λ.

Thus, $\varphi^{nn} \notin \Lambda$, whence $\varphi^n \notin \Lambda^n$. On the other hand, $\varphi^n = \mathtt{ND}_k \in \mathsf{InqL}$. So, $\Lambda^n \neq \mathsf{InqL}$. □

Lemma 3.4.11. For any intermediate logic Λ, if $\Lambda^n = \mathsf{InqL}$, then $\Lambda \subseteq \mathsf{ML}$.

Proof. We know that Λ is always included in Λ^n, so if $\Lambda^n = \mathsf{InqL}$ we have $\Lambda = \mathrm{Sch}(\Lambda) \subseteq \mathrm{Sch}(\Lambda^n) = \mathrm{Sch}(\mathsf{InqL}) = \mathsf{ML}$, where the first equality uses the fact that Λ is closed under uniform substitution. □

Note that as a nice corollary of theorem 3.4.9 we can derive easily a well-known result due to Maksimova.

Corollary 3.4.12. If $\Lambda \supseteq \mathsf{ND}$ is a logic with the disjunction property, then $\Lambda \subseteq \mathsf{ML}$. In particular, ML is a *maximal* logic with the disjunction property.

Proof. According to theorem 3.2.39, if $\Lambda \supseteq \mathsf{ND}$ is a logic with the disjunction property, then $\Lambda^n = \mathsf{InqL}$ and thus, by the previous theorem, $\Lambda \subseteq \mathsf{ML}$. □

It is also easy to strengthen this most general version of the completeness theorem to a *strong completeness* result.

Corollary 3.4.13 (Strong completeness of axiomatizations of InqL)**.** Let Λ be an intermediate logic with $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$. For any set of formulas Θ and any formula φ,

$$\Theta \models_{\mathsf{InqL}} \varphi \iff \Theta \models_{\Lambda^n} \varphi$$

Proof. The soundness direction is obvious since $\Lambda \subseteq \mathsf{ML} \subseteq \mathsf{InqL}$, $\neg\neg p \to p \in \mathsf{InqL}$ and the set of formulas supported by a state is closed under modus ponens. For the completeness direction proceed exactly as in the proof of corollary 3.2.33 concerning the strong completeness of KP^n. □

3.4.4 More on Medvedev logic

The idea to interpret propositional formulas as finite problems goes back to Kolmogorov's approach to intuitionistic logic. Based on Kolmogorov's interpretation, Medvedev (1962) developed a formal *finite problem semantics*, and in (Medvedev, 1966) he showed that the associated logic could be characterized in terms of Kripke models as the logic of the class **Med**.

The quest for an axiomatization of ML did not produce significant results until Maksimova *et al.* (1979) proved that ML is not finitely axiomatizable and indeed not axiomatizable with a finite number of propositional letters. The question of whether ML admits a recursive axiomatization (equivalently, of whether ML is decidable) is a long-standing open problem.

This makes the results we just established rather interesting. For, in the first place we have seen that $\mathsf{ML} = \mathrm{Sch}(\mathsf{InqL}) = \mathrm{Sch}(\mathsf{KP}^n) = \mathrm{Sch}(\mathsf{ND}^n)$, which means that both systems ND^n and KP^n give a sort of recursive pseudo-axiomatization of ML: they derive 'slightly' more formulas than those in ML, but if we restrict our attention to the schematic theorems, then we have precisely Medvedev's logic.

In the second place, corollary 3.4.6 provides a connection between Medvedev's logic, Inquisitive Logic, and other well-understood intermediate logics such as the Kreisel-Putnam logic, which might pave the way for new attempts to solve the decidability problem for ML.

For instance—since InqL is decidable—if it were possible to find a finite bound b for the maximum number k of disjuncts that we need to use in order to falsify a non-schematically valid formula φ (possibly depending on the number of propositional letters in φ) then ML would be decidable. For, to determine whether $\varphi \in \mathsf{ML}$ it would then suffice to check whether the formula $\varphi(\bigvee_{1\leq i\leq k} p^i_1, \dots, \bigvee_{1\leq i\leq k} p^i_n)$ is in InqL for all $k \leq b$, and this procedure can be performed in a finite amount of time.

We will show that at least for formulas containing a single proposition letter p, such a bound exists and equals 2. As a consequence, the one-letter fragment of ML is decidable.

This was already known, since it was known (Medvedev, 1966) that the one-letter fragment of Medvedev logic coincides with Scott logic, axiomatized by $((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p$, and Scott logic is known to be decidable (see Chagrov and Zakharyaschev, 1997, theorem 11.58); but our argument is new and could perhaps be generalized.

Theorem 3.4.14. For any formula $\varphi(p)$ with only one propositional letter, the following are equivalent:

1. $\varphi(p) \in \mathsf{ML}$
2. $\varphi(\neg p \vee \neg q) \in \Lambda$ where Λ is any intermediate logic with $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$
3. $\varphi(p \vee q) \in \mathsf{InqL}$

The equivalence (2) $\iff$ (3) is an immediate consequence of the characterization theorem (theorem 3.4.9) stating that if $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$ then $\Lambda^n = \mathsf{InqL}$. The implication (1) $\Rightarrow$ (3) follows from theorem 3.4.5 stating that $\mathsf{ML} = \mathrm{Sch}(\mathsf{InqL})$. So it just remains to prove that one of (2) or (3) implies (1). In order to do so, we first need some definitions and lemmas.

Definition 3.4.15. The Scott formula, or Scott axiom, denoted $\mathtt{S}$ is the formula:

$$\mathtt{S} \qquad ((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p$$

Lemma 3.4.16. $\mathtt{S} \in \mathsf{ML}$.

Proof. Towards a contradiction, suppose there were a Medvedev model M where $\mathtt{S}$ does not hold. Then there should be a point a point S with $M, S \Vdash (\neg\neg p \to$

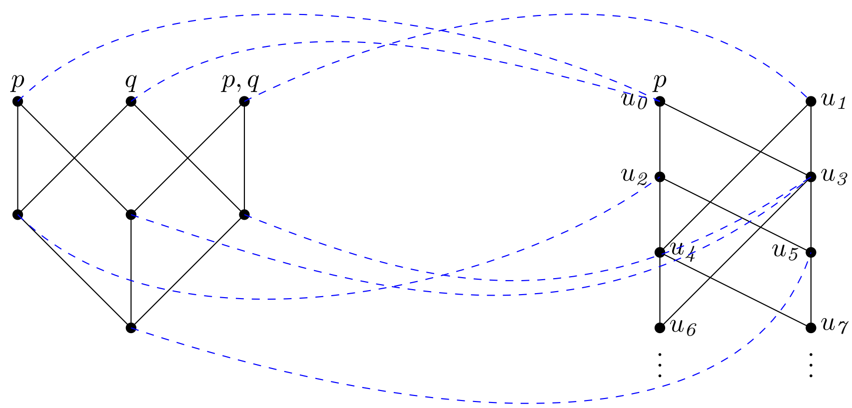


Figure 3.2: On the left, a model $\mathcal{M}_3$ on the Medvedev frame $\wp(3)$. On the right, the Rieger-Nishimura ladder $\mathcal{U}$. The dashed lines represent a frame-morphism η from $\mathcal{M}_3$ to $\mathcal{U}$ such that $\mathcal{M}, w \Vdash \neg p \vee \neg q$ iff $\mathcal{U}, \eta(w) \Vdash (p)$.

$p) \to p \vee \neg p$ but $M, S \nVdash \neg p$ and $M, S \nVdash \neg\neg p$. Recalling that on finite frame a negation holds at a point iff it holds at all its endpoints, this means that there must exist elements $s, t \in S$ with $M, \{s\} \Vdash p$ but $M, \{t\} \Vdash \neg p$.

Now the only successor of the point $\{s, t\}$ where $\neg\neg p$ holds is $\{s\}$, and there also p holds. Thus, $M, \{s, t\} \Vdash \neg\neg p \to p$. However, $M, \{s, t\} \nVdash p$ because $\{t\}$ is a successor of $\{s, t\}$ validating $\neg\neg p$, and $M, \{s, t\} \nVdash \neg p$ because $\{s\}$ is a successor validating p. Hence $M, \{s, t\} \nVdash p \vee \neg p$.

But since s and t are in S we have $\{s, t\} \subseteq S$, that is, $\{s, t\}$ is a successor of S, and therefore we cannot have $M, S \Vdash (\neg\neg p \to p) \to p \vee \neg p$, which contradicts our assumptions. ☐

Lemma 3.4.17. If a formula $\varphi(p)$ containing only the propositional letter p is not in ML, then $\mathcal{U}, u_5 \nVdash \varphi(p)$, where $\mathcal{U}$ is the Rieger-Nishimura ladder in figure 3.2.

Proof. Recall the crucial property of the Rieger-Nishimura ladder (see Chagrov and Zakharyaschev, 1997, sections 8.6 and 8.7): any intuitionistic Kripke model for one letter p can be p-morphically mapped (in a unique way) into $\mathcal{U}$.

Now suppose $\varphi(p) \notin \mathsf{ML}$: then there must be a point w in a Medvedev model M for the letter p with $M, w \nVdash \varphi(p)$. Now let η be a (the) p-morphism from M into $\mathcal{U}$: since satisfaction is invariant under p-morphisms, $\mathcal{U}, \eta(w) \nVdash \varphi(p)$.

Now, it is straightforward to check that $\mathcal{U}, u_4 \nVdash \mathtt{S}$. But for each $i > 5$, u_4 is a successor of u_i and therefore $\mathcal{U}, u_i \nVdash \mathtt{S}$ either.

On the other hand, since M is a Medvedev model according to the previous lemma we have $M, w \Vdash \mathtt{S}$, so also $\mathcal{U}, \eta(w) \Vdash \mathtt{S}$; and since we have seen that $\mathtt{S}$ is not forced at u_i if $i = 4$ or $i > 5$, it must be $\eta(w) = u_i$ for $i = 5$ or $i < 4$.

If $\eta(w) = u_5$, then our claim is true. If on the other hand $\eta(w) = u_i$ for $i < 4$, then $\eta(w)$ is a successor of u_5, and thus by persistency again $\mathcal{U}, u_5 \nVdash \varphi(p)$. So in any case $\varphi(p)$ is not forced at point u_5, which is what we had to show. □

Next, we need to prove a simple generalization of the invariance of satisfaction under p-morphisms.

Lemma 3.4.18. Consider Kripke models M, M' and formulas $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n$. Let η be a map from M to M' that satisfies the forth and the back condition (i.e., that is a frame p-morphism) and such that for any point w in M and any $1 \leq i \leq n$, $M, w \Vdash \varphi_i \iff M', \eta(w) \Vdash \psi_i$.

Then for any formula $\chi(p_1, \ldots, p_n)$ whose propositional letters are among $p_1, \ldots, p_n$ and for any point w in M we have

$$M, w \Vdash \chi(\varphi_1, \ldots, \varphi_n) \iff M', \eta(w) \Vdash \chi(\psi_1, \ldots, \psi_n)$$

Proof. Straightforward by induction on the structure of χ. The atomic case amounts to the assumption that for each $1 \leq i \leq n$ it is $M, w \Vdash \varphi_i \iff M', w \Vdash \psi_i$. The inductive steps for $\bot, \vee, \wedge$ are trivial, and the one for $\rightarrow$ uses the forth and back condition on η to trigger the induction hypothesis, just like for p-morphisms. □

Lemma 3.4.19. Let $\mathcal{M}_3$ be the model on the frame $\wp(3)$ displayed in figure 3.2 and let r be the root of that model. For any formula $\varphi(p)$ containing only the letter p,

$$\mathcal{U}, u_5 \Vdash \varphi(p) \iff \mathcal{M}_3, r \Vdash \varphi(\neg p \vee \neg q)$$

Proof. Consider the map η depicted in figure 3.2. It is immediate to check that for any point w in $\mathcal{M}_3$ we have $\mathcal{M}_3, w \Vdash \neg p \vee \neg q \iff \mathcal{U}, \eta(w) \Vdash p$.

It is equally straightforward to check that η satisfies the forth and back conditions, that is, η is a frame p-morphism from $\wp(3)$ to the Rieger-Nishimura frame. Thus, noting that $\eta(r) = u_5$, our claim follows from the previous lemma. □

We are now ready to complete the proof of theorem 3.4.14

Proof of theorem 3.4.14, concluded. It order to prove the theorem, it remained to show that (2) implies (1). Choosing $\Lambda = \mathsf{ML}$ in (2), it suffices to show that $\varphi(\neg p \vee \neg q) \in \mathsf{ML}$ implies $\varphi(p) \in \mathsf{ML}$.

Contrapositively, suppose $\varphi(p) \notin \mathsf{ML}$. Then, according to lemma 3.4.17, $\mathcal{U}, u_5 \nVdash \varphi(p)$; so, according to lemma 3.4.19, $M_3, r \nVdash \varphi(\neg p \vee \neg q)$. But M_3 is a Medvedev model, since it is based on the powerset frame $\wp(3)$: so, $\varphi(\neg p \vee \neg q) \notin \mathsf{ML}$. □

As a corollary, we immediately get the decidability of the one-letter fragment of Medvedev logic.

Corollary 3.4.20 (Decidability of the one-letter fragment of ML)**.** The the problem of determining whether a formula $\varphi(p)$ in one propositional letter belongs to ML is decidable.

Proof. We have just seen a number of ways to decide whether a formula $\varphi(p)$ is in ML. For instance:

1. check whether $\mathcal{M}_3, r \Vdash \varphi(\neg p \vee \neg q)$;
2. check whether $\varphi(p \vee q) \in \mathsf{InqL}$;
3. check whether $\varphi(\neg p \vee \neg q) \in \mathsf{KP}$

□

3.5 Independence of the connectives

In section 2.5 we saw that the $\{\neg, \vee\}-$fragment of the language suffices to express all inquisitive meanings, and therefore that conjunctions and implications (except for negations) can always be eliminated from a formula preserving its meaning.

Does this mean that conjunction and implication are superfluous in inquisitive semantics? Not really. For, in this section we will show that in inquisitive semantics, no connective can be defined in terms of the others.

There is, for instance, no way to simulate the way the conjunction operator works by means of $\vee$ and $\neg$ in a uniform way. Given *specific* formulas φ and ψ, we can always find a way of expressing $\varphi \wedge \psi$ without conjunction. But there is no way to do so *schematically*, that is, without knowing what φ and ψ are.

For instance, exactly like in classical logic, $p \wedge q \equiv \neg(\neg p \vee \neg q)$. But, for instance $p \wedge (q \vee r)$ is *not* equivalent to $\neg(\neg p \vee \neg(q \vee r))$, as the latter is an assertion and the former is not. Still, the conjunction can be eliminated by translating $p \wedge (q \vee r)$ as $\neg(\neg p \vee \neg q) \vee \neg(\neg p \vee \neg r)$. This should give an intuition that the way conjunction is replaced crucially depends on the number of possibilities for the conjuncts. The same applies to implication.

Definition 3.5.1. We say that a formula $\xi(p_1, \dots, p_n)$ *defines* an n-ary connective $\circ$ in inquisitive semantics in case $\circ(\alpha_1, \dots, \alpha_n) \equiv_{\mathsf{InqL}} \xi(\alpha_1, \dots, \alpha_n)$ for all formulas $\alpha_1, \dots, \alpha_n$. We say that $\circ$ is definable in terms of a set C of connectives in case there exists a formula $\xi(p_1, \dots, p_n)$ containing only the connectives in C that defines $\circ$.

The following lemma shows that definability in inquisitive semantics amounts to definability in Medvedev logic.

Lemma 3.5.2. The formula $\xi(p_1, \dots, p_n)$ defines $\circ$ in inquisitive semantics if and only if $\circ(p_1, \dots, p_n) \equiv_{\mathsf{ML}} \xi(p_1, \dots, p_n)$.

Proof. By the deduction theorem, $\xi(p_1, \ldots, p_n)$ defines the connective $\circ$ iff the formula $\circ(\alpha_1, \ldots, \alpha_n) \leftrightarrow \xi(\alpha_1, \ldots, \alpha_n)$ is in InqL for all formulas $\alpha_1, \ldots, \alpha_n$; clearly, this is equivalent to saying that $\circ(p_1, \ldots, p_n) \leftrightarrow \xi(p_1, \ldots, p_n)$ must be in Sch(InqL), which by theorem 3.4.5 equals ML. □

The following propositions show that in inquisitive semantics (and thus in Medvedev logic as well) all the connectives are independent, i.e. non-interdefinable.

Proposition 3.5.3. $\bot$ is not definable in terms of $\wedge, \vee$ and $\rightarrow$.

Proof. Let i_1 be the index making all propositional letters true. It is utterly straightforward to prove by induction that $i_1 \models \varphi$ for any formula φ built up with the connectives $\wedge, \vee$ and $\rightarrow$ only.

Therefore, by the classical behaviour of singletons, the state $\{i_1\}$ supports any formula not containing $\bot$, while it does not support $\bot$. Therefore, $\bot$ is not equivalent to any formula containing only $\wedge, \vee$ and $\rightarrow$, and thus not definable in terms of those connectives. □

Proposition 3.5.4. $\vee$ is not definable in terms of $\wedge, \rightarrow$ and $\bot$.

Proof. According to corollary 2.1.22, any disjunction-free formula is an assertion. Since $p \vee \neg p$ is *not* an assertion, it cannot be equivalent to any disjunction-free formula. Hence, disjunction is not definable in terms of the other connectives. □

Proposition 3.5.5. $\wedge$ is not definable in terms of $\vee, \rightarrow$ and $\bot$.

Proof. Towards a contradiction, suppose $\wedge$ were definable in terms of $\vee, \rightarrow$ and $\bot$. Let $\xi(p, q)$ be a formula of minimal length that defines $\wedge$. Certainly this formula cannot be any of p, q and $\bot$. Therefore, it must be either a disjunction or an implication.

Observe that both in classical and intuitionistic logic, the conjunction $p \wedge q$ entails either $\psi(p, q)$ or $\neg\psi(p, q)$ for any formula $\psi(p, q)$ containing only the propositional letters p and q, and which one of the two is entailed is determined truth-functionally according to the classical rules. Hence, the same is true in Medvedev logic. Now let us reason by cases.

- Suppose $\xi(p, q)$ is a disjunction $\varphi(p, q) \vee \psi(p, q)$. It cannot be the case that $p \wedge q \models_{\mathsf{ML}} \neg\varphi$ and $p \wedge q \models_{\mathsf{ML}} \neg\psi$; for, otherwise $p \wedge q$ would entail $\neg\varphi \wedge \neg\psi$ and therefore also the equivalent formula $\neg(\varphi \vee \psi)$, that is, we would have $p \wedge q \models_{\mathsf{ML}} \neg\xi$. But this is a contradiction, since by assumption ξ defines $\wedge$ and therefore according to lemma 3.5.2 we have $p \wedge q \equiv_{\mathsf{ML}} \xi$, so ML would be inconsistent.

  Hence, by our previous observation, at least one of $p \wedge q \models_{\mathsf{ML}} \varphi$ and $p \wedge q \models_{\mathsf{ML}} \psi$ must hold, and without loss of generality we may assume that the former is the case.

  On the other hand, $\varphi \models_{\mathsf{ML}} \varphi \vee \psi = \xi$ and by lemma 3.5.2 $\xi \equiv_{\mathsf{ML}} p \wedge q$, so $\varphi \models_{\mathsf{ML}} p \wedge q$. In conclusion we have $p \wedge q \equiv_{\mathsf{ML}} \varphi$, which means that φ defines $p \wedge q$.

But this is a contradiction, since we assumed that $\varphi \vee \psi$ was a formula *of minimal length* defining conjunction. So ξ cannot be a disjunction.

- Suppose $\xi(p, q)$ is an implication $\varphi(p, q) \to \psi(p, q)$. If $p \wedge q \models_{\mathsf{ML}} \varphi$, then since we are assuming that $p \wedge q \equiv_{\mathsf{ML}} \varphi \to \psi$ we also have $p \wedge q \models_{\mathsf{ML}} \psi$. But on the other hand, $\psi \models_{\mathsf{InqL}} \varphi \to \psi \equiv_{\mathsf{ML}} p \wedge q$, so $p \wedge q \equiv_{\mathsf{ML}} \psi$, which means that ψ defines $p \wedge q$. But this is impossible, since by assumption $\varphi \to \psi$ was a formula *of minimal length* defining $p \wedge q$.

  So $p \wedge q$ cannot entail φ, and thus by the previous observation it must be $p \wedge q \models_{\mathsf{ML}} \neg\varphi$. On the other hand, $\neg\varphi \models_{\mathsf{ML}} \varphi \to \psi \equiv_{\mathsf{ML}} p \wedge q$, so $p \wedge q \equiv_{\mathsf{ML}} \neg\varphi$, which means that $\neg\varphi(p, q)$ defines conjunction.

  But this cannot be the case, since we know that $\neg\varphi(\alpha_1, \alpha_2)$ is an assertion for all α_1, α_2, whereas $\alpha_1 \wedge \alpha_2$ is not an assertion in general: for instance, $?p \wedge ?p \equiv ?p$ is not an assertion.

  So ξ cannot be an implication either, and we have the required contradiction.

□

Proposition 3.5.6. $\to$ is not definable in terms of $\vee, \wedge$, $\bot$ and even $\neg$ if taken as primitive.

Proof. By lemma 3.5.2, it suffices to show that $\to$ is not definable in terms of the other connectives in ML.

Now, Diego's theorem (see Chagrov and Zakharyaschev, 1997, theorem 5.37) asserts that in intuitionistic logic there are only finitely many non-equivalent implication-free formulas over a finite set of propositional letters is finite, even when negation is also allowed as a primitive connective. Since equivalence in IPL implies equivalence in ML the same holds in Medvedev logic.

If implication were definable in terms of $\vee, \wedge, \bot$ and $\neg$ in ML, then any formula would be equivalent to an implication-free formula, and thus there would be only finitely many non-equivalent formulas over a finite set of propositional letters; that is, to put it in technical terms, ML would be locally tabular.

However, ML is *not* locally tabular: for, it is a well known fact that $\mathsf{ML} \subseteq \mathsf{KC}$, where KC is the intermediate logic axiomatized by $\neg p \vee \neg\neg p$; and it is equally well-known (Jankov, 1963, see) that KC is *not* locally tabular, whence ML cannot be locally tabular either. □

Chapter 4

The inquisitive hierarchy

As we said in the introduction, the first implementation of inquisitive semantics —due to Groenendijk (2008c) and Mascarenhas (2008)— was different from the one considered here: formulas were evaluated with respect to *ordered pairs* of indices rather than w.r.t. arbitrary *sets* of indices. The intuition underlying that semantics came from the tradition of Groenendijk's logic of interrogation (Groenendijk, 1999), where the meaning of a formula consisted in a relation that connects two models in case the formula is *indifferent* between them. We will refer to that system as the *pair semantics*, and to the system considered in chapter 2 as the *generalized semantics*.

In this chapter we observe that the pair semantics is the particular case of our generalized semantics in which only states of size at most two are allowed. Thus, the pair semantics can be viewed as an element in a sequence of semantics obtained by restricting the generalized semantics to states of cardinality at most n; we show that these semantics give rise to a hierarchy of strictly shrinking logics whose limit case is InqL. We then provide a uniform axiomatization of the layers of the hierarchy. Finally, we show that all the restricted semantics have shortcomings that are only avoided by the generalized semantics, which we argue to be the system that correctly matches our desiderata.

4.1 Generalizing the pair semantics

In the pair semantics, support is a relation between ordered pairs of indices and formulas defined as follows.

Definition 4.1.1 (Pair semantics)**.** For any indices i and j,

1. $\langle i, j\rangle \models p$ iff $i \models p$ and $j \models p$
2. $\langle i, j\rangle \not\models \bot$
3. $\langle i, j\rangle \models \varphi \wedge \psi$ iff $\langle i, j\rangle \models \varphi$ and $\langle i, j\rangle \models \psi$

4. $\langle i,j\rangle \models \varphi \vee \psi$ iff $\langle i,j\rangle \models \varphi$ or $\langle i,j\rangle \models \psi$
5. $\langle i,j\rangle \models \varphi \to \psi$ iff for all $k,h \in \{i,j\}$, if $\langle k,h\rangle \models \varphi$ then $\langle k,h\rangle \models \psi$

It is immediate to see that for any indices v, w and any formula φ, $\langle v,w\rangle \models \varphi$ amounts precisely to our $\{v,w\} \models \varphi$. In this sense, the pair semantics can be seen as a fragment of the generalized semantics, namely the fragment dealing only with states of size one (in case $v = w$) or two (in case $v \neq w$).

In other words, just like it contains a copy of classical semantics given by states of size one, generalized inquisitive semantics contains a copy of the pair semantics, given by the states of size one or two.

This restriction on the cardinality of states gives rise to a logic InqL_2, much stronger than InqL, which has been studied and axiomatized by Mascarenhas (2008).

Definition 4.1.2 (LV)**.** LV is the intermediate logic axiomatized by the following formulas:

δ_2 $\quad p \vee (p \to q \vee \neg q)$

γ_2 $\quad (p \to q) \vee (q \to p) \vee ((p \to \neg q) \wedge (q \to \neg p))$

It is easy to see that in fact LV is the logic of the two-fork frame $\bigvee$, that is, $(\wp(2) - \{\emptyset\}, \supseteq)$.

Theorem 4.1.3 (Mascarenhas, 2009)**.** $\mathsf{InqL}_2 = \mathsf{LV}^n$.

Now, in general, we may consider the logic InqL_n arising from restricting inquisitive semantics by allowing only states of cardinality at most n. As we shall see shortly, doing so gives rise to a hierarchy of strictly shrinking logics whose limit is InqL. Throughout this section we assume a countably infinite set of propositional letters $\mathcal{P} = \{p_i \mid i \in \omega\}$.

Definition 4.1.4 (Inquisitive Hierarchy)**.**
For $k \in \omega$, define $\mathsf{InqL}_k = \{\varphi \mid s \models \varphi$ for any state s with $|s| \leq k\}$.

The only state of cardinality zero is $\emptyset$, which supports any formula, so InqL_0 is the inconsistent logic. Moreover, since singleton states behave like indices, InqL_1 is classical logic (the addition of the empty state clearly does not make a difference). And, according to the previous observations, InqL_2 is the logic arising from the pair semantics. The following lemma suffices to prove that InqL is indeed the limit of the inquisitive hierarchy.

Lemma 4.1.5. Any formula that is not inquisitively valid is falsifiable on a state of finite cardinality.

Proof. Take $\varphi \notin \mathsf{InqL}$. We know by corollary 2.1.14 that φ has only finitely many possibilities, say $\pi_1, \ldots, \pi_n$. Now since φ is not valid, none of these coincides with the full board $\mathcal{I}$: therefore, for $1 \leq k \leq n$ we can take an index $i_k \notin \pi_k$. Then the finite state $\{i_1, \ldots, i_n\}$ is not included in any possibility for φ and therefore according to proposition 2.1.10 it does not support φ. $\square$

Corollary 4.1.6. $\mathsf{InqL} = \bigcap_{k\in\omega}\mathsf{InqL}_k$

We will now define, for each $k \in \omega$, a formula δ_k that characterizes the class of intuitionistic Kripke frames of depth at most k.

We will then show, first, that for any $k \in \omega$, δ_k is in $\mathsf{InqL}_k - \mathsf{InqL}_{k+1}$, which means that the inquisitive hierarchy is a sequence of strictly shrinking logics, and second, that for any $k \in \omega$, adding all substitution instances of δ_k to an axiomatization of InqL yields an axiomatization of InqL_k.

Definition 4.1.7. The formulas δ_k, $k \in \omega$ are defined inductively as follows.

- $\delta_0 := \bot$
- $\delta_{k+1} := p_{k+1} \vee (p_{k+1} \to \delta_k)$

An easy induction suffices to check that, indeed, for any natural k, δ_k is valid on a frame F iff the depth[1] of F is at most k.

Proposition 4.1.8. For any natural k, $\delta_k \in \mathsf{InqL}_k$ but $\delta_k \notin \mathsf{InqL}_{k+1}$.

Proof. First let us remark that for any finite non-empty state s, the depth of the subframe $(F_I)_s$ generated by the point s in the frame F_I of the Kripke model for inquisitive logic is equal to $|s|$. This can be checked by an easy induction.

Now, first consider any state s with $|s| \leq k$: we have to check that $s \models \delta_k$. We may assume $s \neq \emptyset$, as our claim is otherwise trivial. Now as the depth of the frame $(F_I)_s$ is $|s| \leq k$, the formula δ_k is valid on $(F_I)_s$, whence in particular $((F_I)_s, V_I), s \Vdash \delta_k$. But then, since Kripke satisfaction is invariant under taking generated submodels, we have $M_I, s \Vdash \delta_k$, which by proposition 2.2.2 amounts to $s \models \delta_k$. This shows that δ_k is supported by any state of size at most k. Hence, $\delta_k \in \mathsf{InqL}_k$.

Second, we have to show that $s \not\models \delta_k$ for some state s with $|s| = k + 1$. We shall proceed by induction on k. For $k = 0$, simply take $s_0 = \{w\}$ where w is the index making all proposition letters true.

Now, inductively, we can assume that we have a state s_k such that $|s_k| = k+1$, $|s_k| \not\models \delta_k$, and *moreover* all indices in s_k make p_j true for all $j > k$. Now simply let $s_{k+1} := s_k \cup \{w\}$ where w is the index making a letter p_j true iff $j \geq k + 2$. It is immediate to check that $s_{k+1} \not\models \delta_{k+1}$, and since $|s_{k+1}| = k + 2$ and all indices in s_{k+1} make p_j true for all $j > k + 1$, the inductive step is complete. $\square$

This proves that the inquisitive hierarchy is indeed a hierarchy of strictly shrinking logics: for any $k \in \omega$, $\mathsf{InqL}_k \supsetneq \mathsf{InqL}_{k+1}$. We now turn to the task of axiomatizing the logics in the hierarchy.

[1] The *depth* of a finite frame $F = (W, \leq)$ is defined as the least natural k such that in F there are no chains of cardinality k.

4.2 Axiomatizing the inquisitive hierarchy

Definition 4.2.1 (Δ_k)**.** For any natural k we denote by Δ_k the intermediate logic axiomatized by the formula δ_k.

Theorem 4.2.2 (Axiomatization of the inquisitive hierarchy)**.** Let Λ be an intermediate logic with $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$. Then $\mathsf{InqL}_k = (\Lambda + \Delta_k)^n$.

In other words, the theorem states that InqL_k is soundly and completely axiomatized by a derivation system having modus ponens as derivation rule, and the following axioms:

1. Λ, or axioms for Λ in schematic form;
2. Δ_k, or δ_k as an axiom scheme;
3. $\neg\neg p \to p$ for $p \in \mathcal{P}$.

where Λ is any logic between ND and ML. For instance, if we choose $\Lambda = \mathsf{KP}$, then the theorem says that we can take our derivation system to have, as axioms, all substitution instances of the formulas $\mathtt{KP}$ and δ_k, plus atomic double negation axioms.

Proof. For the soundness direction, since the set InqL_k is closed under modus ponens and contains Λ and the atomic double negation axioms, it suffices to check that any substitution instance ${\delta_k}^*$ of δ_k is in InqL_k. Consider a state s with $0 < |s| \leq k$ (the case $|s| = 0$ is trivial): then the generated subframe $(F_I)_s$ of s in the frame F_I has depth at most k, so δ_k is valid on $(F_I)_s$, and since the logic of a frame is closed under uniform substitution, ${\delta_k}^*$ is valid on $(F_I)_s$ as well. In particular we have $((F_I)_s, V_I), s \Vdash {\delta_k}^*$, which by invariance of Kripke-satisfaction under generated submodels yields $M_I, s \Vdash {\delta_k}^*$ and thus $s \models {\delta_k}^*$.

For completeness, suppose $\varphi \notin (\Lambda + \Delta_k)^n$, that is, assume that $\Delta_k \nvdash_{\Lambda^n} \varphi$. By the strong completeness of Λ^n (corollary 3.4.13) there must be a state s such that $s \models \Delta_k$ but $s \not\models \varphi$. Now in order to conclude $\varphi \notin \mathsf{InqL}_k$ it suffices to show that $s \models \Delta_k$ implies $|s| \leq k$. This is the content of the following lemma. ☐

Lemma 4.2.3. For any $k \in \omega$, if $|s| > k$ then there is a substitution instance ${\delta_k}^*$ of the formula δ_k such that $s \not\models {\delta_k}^*$

Proof. We will proceed by induction on k. The case for $k = 0$ is trivial: $\delta_0 = \bot$ is not supported by any non-empty state.

Now, assume our claim holds for a number k and consider a state s with $|s| > k + 1$. We may assume that s is finite: if not, just replace it by a finite substate $s' \subseteq s$ with $|s'| > k + 1$: then by persistence, once we find an instance of δ_k which is not supported by s', this cannot be supported by s either.

Fix an index $w \in s$: exploiting the fact that w must differ from any other $w' \in s$ on some letter and that s contains only finitely many indices, we can easily find a formula γ such that $w \models \gamma$ but $w' \not\models \gamma$ for any $w' \neq w$ in s.

Now, since $|s - \{w\}| > k$, by induction hypothesis there is a substitution instance ${\delta_k}^*$ of δ_k such that $s - \{w\} \not\models \delta_k$. Since δ_k contains only the variables $p_1, \ldots, p_k$, the substitution we need in order to get ${\delta_k}^*$ from δ_k does not concern the variable p_{k+1}, for which we are free to choose a substitute: thus, the formula ${\delta_{k+1}}^* := \neg\gamma \vee (\neg\gamma \to {\delta_k}^*)$ is indeed a substitution instance of $\delta_{k+1} = p_{k+1} \vee (p_{k+1} \to \delta_k)$.

Now $s \not\models \neg\gamma$, because $w \in s$ and $w \not\models \neg\gamma$; on the other hand, $s \not\models \neg\gamma \to {\delta_k}^*$, because $s - \{w\} \not\models \delta_k$ while $s - \{w\} \models \neg\gamma$ by proposition 2.1.6. Thus, $s \not\models {\delta_{k+1}}^*$ and we are done: we have shown that for any state s with $|s| > k + 1$ there is a substitution instance of δ_{k+1} which is not supported by s.

□

Note that, proceeding in exactly the same way, we could in fact have shown that, for $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$, the system $(\Lambda + \Delta_k)^n$ provides a *strongly* complete axiomatization of InqL_k, in the sense that it captures the notion of entailment $\models_{\mathsf{InqL}_k}$ that results from restricting inquisitive semantics to states of size at most k. For any set Θ and any formula φ,

$$\Theta \models_{\mathsf{InqL}_k} \varphi \iff \Theta \vdash_{(\Lambda+\Delta_k)^n} \varphi$$

4.3 A Plea for the Generalized Semantics

We have seen that in the pair semantics, formulas are evaluated with respect to ordered pairs of indices. The possibilities for a formula φ are then defined as maximal states such that any pair of indices in that state supports φ. This notion of possibilities subtly differs from the one that the generalized semantics gives rise to. Thus, the pair semantics and the generalized semantics yield a different notion of meaning. In this section, we compare these two notions of meaning and argue that the differences speak in favour of the generalized semantics.

In order to make such a judgment, we must first of all determine a suitable criterion for comparison. In order to do so, we return to one of the main sources of inspiration behind inquisitive semantics, the 'Gricean picture of disjunction'. Grice (1989, page 68) gives the following picture of the use of disjunction:

> A standard (if not the standard) employment of 'or' is in the specification of possibilities (one of which is supposed by the speaker to be realized, although he does not know which one), each of which is relevant in the same way to a given topic. 'A or B' is characteristically employed to give a partial answer to some [wh]-question, to which each disjunct, if assertible, would give a fuller, more specific, more satisfactory answer.

This picture has played an important role in the development of inquisitive semantics (cf. Groenendijk, 2008b), and indeed, a disjunction $p \vee q$ is assigned a meaning consisting of two alternative possibilities, $|p|$ and $|q|$.

Now, the Gricean picture of disjunction is of course not intended to apply only to disjunctions with *two* disjuncts. It applies just as well to disjunctions with three or more disjuncts. For instance, the idea is that a triple disjunction $p \vee q \vee r$ is used to specify three possibilities, $|p|$, $|q|$ and $|r|$. One criterion, then, for comparing different implementations of inquisitive semantics, is that the Gricean picture of disjunction should be captured for disjunctions of arbitrary length.

This is indeed what the generalized semantics does, unlike any other element of the inquisitive hierarchy. Let us consider the pair semantics in particular. This semantics assigns to $p \vee q \vee r$ the three possibilities $|p|$, $|q|$ and $|r|$, but also an additional possibility $t = \{111, 110, 101, 011\}$ (since every *pair* of indices in t supports $p \vee q \vee r$).[2]

More generally, a disjunction $p_1 \vee \ldots \vee p_{n+1}$ will be problematic for any element of the inquisitive hierarchy that looks only at states of size at most n. The Gricean picture is only captured in full generality if states of arbitrary size are taken into account.

Now let us consider another criterion, which concerns the intuition that the possibilities for a sentence φ correspond to the alternative ways in which φ may be resolved. Again, upon close examination, it turns out that this intuition is captured by the generalized semantics, but not by any other element of the inquisitive hierarchy.

It is easy to see that the pair semantics can only disagree with the generalized semantics about the meaning of a formula φ in case one of the possibilities that the pair semantics assigns to φ, call it t, does not support φ; so, according to proposition 2.3.6, φ must be informative or inquisitive in t; but it cannot be informative, since both semantics yield the same, classical treatment of information, so it must be inquisitive: for instance, $p \vee q \vee r$ is inquisitive in the state $\{111, 110, 101, 011\}$, which is a possibility according to the pair semantics.

This means that the issue raised by φ is still open in t, and t cannot therefore constitute a possible way of resolving φ. Still, the pair semantics considers t to be a possibility for φ. We conclude that the pair semantics does not yield the intended notion of possibilities, and the same point can easily be made for any other element of the inquisitive hierarchy.

On the other hand, if we read support as giving conditions for a formula to be resolved in a state, then the possibilities we get in the generalized semantics are *by definition* maximal states in which the formula is resolved, and as such they mirror the alternative ways to resolve a formula. This —perhaps obscure— last passage will be made clearer and more formal in chapter 6 with the introduction of the notion of *resolutions* of a formula.

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[2] It does not help trying to justify the presence of t as ‘the possibility that at least two’; otherwise there would be no reason why we should not also have a ‘possibility that at least two’ for $p \vee q$, which the pair semantics does not give. And this to say nothing of what happens in the case of *four* disjuncts!

Chapter 5

Intermediate logics with negative atoms

In section 3.4.3 we have seen that if $\mathsf{ND} \subseteq \Lambda \subset \mathsf{ML}$, then adding the double negation axiom $\neg\neg p \to p$ for atomic formulas alone and closing under the *modus ponens* rule yields new schematic validities.

In this section we are going to undertake a more general investigation of this issue: given an intermediate logic Λ, what is the set of schematic validities that we get by expanding Λ with atomic double negations? In other words, do we get a stronger logic if restrict ourselves to deal with negative atoms?

According to proposition 3.2.12, the set of formulas we get by expanding a logic Λ with atomic double negations is nothing but the negative variant Λ^n of Λ. So, we can define the operation of *negative closure* that we are going to study as follows.

Definition 5.0.1. For any intermediate logic Λ, its *negative closure* is the set of schematic validities of its negative variant:

$$\Lambda^\nu = \mathrm{Sch}(\Lambda^n)$$

In this notation, the result of combining theorems 3.4.9 and 3.4.5 can be formulated as follows: $\Lambda^n = \mathsf{ML}$ whenever $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$. Moreover, we shall see in a moment that for any intermediate logic Λ we have $(\Lambda^\nu)^n = \Lambda^n$, so in fact the following holds for any Λ:

$$\Lambda^\nu = \mathsf{ML} \iff \mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$$

In section 5.2 we shall look at the effect of this operation on other well-known intermediate logics.

But before coming on to that, in the next section we will recall the notion of a *closure operator* and some basic facts about it; we will then see that the operation of negative closure is indeed a closure operator and draw some general conclusions about the behaviour of $(\ )^\nu$.

5.1 Negative closure of intermediate logics

Definition 5.1.1. A *closure operator* on a partial order $\mathbb{X} = (X, \leq)$ is a map $c : X \to X$ that has the following three properties:

1. Extensivity: for all $x \in X$, $x \leq c(x)$.
2. Monotonicity: for all $x, y \in X$, if $x \leq y$ then $c(x) \leq c(y)$.
3. Idempotence: for all $x \in X$, $c(c(x)) = c(x)$.

The notion of closure operator is standard in order theory. Elements of X such that $c(x) = x$ are usually called *closed*, but in view of our interest in the operator $(\ )^\nu$, it will be convenient to refer to them as *stable* elements. Observe that, by idempotence, the set of stable elements coincides with the image $c[X]$ of X under c.

The facts expressed by propositions 5.1.2 and 5.1.3 are well-known. We will recall the short proof of the former; for the latter, the reader is referred to Burris and Sankappanavar (1981), theorem 5.2.

Proposition 5.1.2. Let $\mathbb{X}$ be a complete lattice and c a closure operator on $\mathbb{X}$. Then any meet of stable elements is stable.

Proof. Let k_i for $i \in I$ be a family of stable elements. Fixed $j \in I$, by monotonicity we have $c(\bigwedge_{i \in I} k_i) \leq c(k_j) = k_j$; and since this holds for all $j \in I$, $c(\bigwedge_{i \in I} k_i) \leq \bigwedge_{i \in I} k_i$. The other inequality is immediate by the extensivity of c. □

Proposition 5.1.3. Let $\mathbb{X}$ be a complete lattice and c a closure operator on $\mathbb{X}$. The set $c[X]$ of stable elements with the induced ordering is a complete lattice, where meet and join are given by:

- $\prod_{i \in I} c(x_i) = \bigwedge_{i \in I} c(x_i)$
- $\coprod_{i \in I} c(x_i) = c(\bigvee_{i \in I} x_i)$

where $\bigwedge$ and $\bigvee$ denote, respectively, the meet and join of $\mathbb{X}$.

A prominent example of closure operator in logic is given by the operation mapping a set of formulas to its deductive closure for a given consequence relation.

The choice of the term *negative closure* for $(\ )^\nu$ is no accident, as the following proposition shows.

Proposition 5.1.4. The operation $(\ )^\nu$ of negative closure is a closure operator on the set of intermediate logics ordered by inclusion.

Proof. We first show that Λ^ν is indeed an intermediate logic for any intermediate logic Λ.

$\mathsf{IPL} \subseteq \Lambda^\nu$. This is immediate, since $\mathsf{IPL} \subseteq \Lambda$ and $\Lambda \subseteq \Lambda^\nu$; the latter inclusion follows from the fact that $\Lambda \subseteq \Lambda^n$ and Λ is closed under uniform substitution.

Closure under modus ponens. Suppose φ and $\varphi \to \psi$ are in Λ^ν. Then given an arbitrary substitution map $(\ )^*$, both φ^* and $(\varphi \to \psi)^* = \varphi^* \to \psi^*$ are in Λ^n, and thus since Λ^n is closed under modus ponens, $\psi^* \in \Lambda^n$. Hence, $\psi \in \Lambda^\nu$.

This shows that Λ^ν is closed under modus ponens.

Closure under uniform substitution. Suppose $\varphi \in \Lambda^\nu$ and consider a substitution instance φ^* of φ. Now, any substitution instance $(\varphi^*)^\star$ of φ^* is also a substitution instance of φ, and therefore it follows from $\varphi \in \Lambda^\nu = \mathrm{Sch}(\Lambda^n)$ that $(\varphi^*)^\star \in \Lambda^n$. This shows that $\varphi^* \in \Lambda^\nu$ and thus that Λ^ν is closed under uniform substitution.

Consistency. If $\bot \in \Lambda^\nu$, then $\bot \in \Lambda^n$, which would mean that $\bot = \bot^n \in \Lambda$. But this is impossible as Λ is an intermediate logic by assumption.

So, Λ^ν can be characterized as the greatest intermediate logic included in the negative variant Λ^n of a logic Λ.

Now let us move on to show that $(\ )^\nu$ is a closure operator. We have already seen that $(\ )^\nu$ is extensive, and monotonicity is immediate from the definition. To see that $(\ )^\nu$ is idempotent, consider an intermediate logic Λ: it is immediate by extensivity that $\Lambda^\nu \subseteq \Lambda^{\nu\nu}$.

For the converse inclusion, it suffices to show that $(\Lambda^\nu)^n \subseteq \Lambda^n$. So, take $\varphi \in (\Lambda^\nu)^n$: this means that $\varphi^n \in \Lambda^\nu = \mathrm{Sch}(\Lambda^n)$. Thus, in particular, $\varphi^{nn} \in \Lambda^n$, which means that $\varphi^{nnn} \in \Lambda$. But since in intuitionistic logic triple negation is equivalent to single negation, φ^{nnn} is equivalent to φ^n, so $\varphi^n \in \Lambda$, which means that $\varphi \in \Lambda^n$. This shows $(\Lambda^\nu)^n \subseteq \Lambda^n$ and completes the proof. □

An intermediate logic that is stable for the negative closure operator will simply be called a *stable logic*. For instance, as a particular case of the above remark that $\Lambda^\nu = \mathsf{ML}$ for $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$ we have that Medvedev logic is stable.

It is well-known (see, for instance, Chagrov and Zakharyaschev, 1997, section 4.1) that intermediate logics ordered by inclusion form a complete lattice, where the meet of a family of logic coincides with the intersection, and the join is the logic axiomatized by their union (i.e., the deductive closure of the union).

Given this observation, the following two facts are particular cases of propositions 5.1.2 and 5.1.3 respectively.

Corollary 5.1.5. The intersection of an arbitrary family of stable logics is stable.

Corollary 5.1.6. Stable logics ordered by inclusion form a complete lattice. The meet of a family of stable logics coincides with the intersection, while the join is the negative closure of the logic axiomatized by the union.

This implies that given intermediate logics Λ_i for $i \in I$, the smallest stable logic containing all of them is the set of schematic validities of the weak logic axiomatized by adding the atomic double negation axioms to $\bigcup_{i\in I}\Lambda_i$.

Another interesting feature of the operation of negative closure lies in the fact that it maps logics with the disjunction property to logics with the disjunction property.

Proposition 5.1.7 (The negative closure operation preserves the disjunction property)**.** If an intermediate logic Λ has the disjunction property, so does Λ^ν.

Proof. We have already remarked (see remark 3.2.5) that if Λ has the disjunction property so does its negative variant Λ^n.

Now, suppose two formulas φ, ψ are both not in Λ^ν. This means that there are two substitution maps $(\ )^*$ and $(\ )^\star$ such that $\varphi^* \notin \Lambda^n$ and $\psi^\star \notin \Lambda^n$.

We can then combine $(\ )^*$ and $(\ )^\star$ into one substitution that falsifies the disjunction $\varphi \vee \psi$ as follows. Let p, q denote two propositional letters that occur in neither φ nor ψ and let $(\ )^+$ be the substitution defined as follows: for any letter r, $r^+ := (p \wedge r^*) \vee (q \wedge r^\star)$. We claim that $(\varphi \vee \psi)^+ \notin \Lambda^n$.

Since $(\varphi \vee \psi)^+ = \varphi^+ \vee \psi^+$, and Λ^n has the disjunction property, in order to conclude $(\varphi \vee \psi)^+ \notin \Lambda^n$ it suffices to show that neither φ^+ nor ψ^+ is in Λ^n.

Consider for instance φ^+: the argument for ψ^+ is then completely symmetrical. To see that φ^+ is not in Λ^n, consider the simple substitution map $(\ )^{[\top/p, \bot/q]}$ that substitutes $\top$ for p, $\bot$ for q and leaves the remaining letters unchanged. For any propositional letter $r \neq p, q$ we have:

$$(r^+)^{[\top/p, \bot/q]} = (\top \wedge r^*) \vee (\bot \wedge r^\star) \equiv_{\mathsf{IPL}} r^*$$

So, recalling that p and q do not occur in φ, we have $(\varphi^+)^{[\top/p, \bot/q]} \equiv_{\mathsf{IPL}} \varphi^*$.

Now, we know that Λ^n is not closed under uniform substitution. However, the substitution $(\ )^{[\top/p, \bot/q]}$ is obviously negative with respect to Λ^n, since in Λ^n the formulas $\top, \bot$ and each atom r are equivalent to their double negation. Therefore, according to proposition 3.2.15, Λ^n is closed under this substitution.

Hence, if φ^+ were in Λ^n, then so would be $(\varphi^+)^{[\top/p, \bot/q]}$; but this is absurd since $(\varphi^+)^{[\top/p, \bot/q]}$ is equivalent to φ^* (in IPL and therefore also in Λ^n) and by assumption $\varphi^* \notin \Lambda^n$. Thus, $\varphi^+ \notin \Lambda^n$.

This proves our claim that $(\varphi \vee \psi)^+ \notin \Lambda^n$, from which it follows immediately that $\varphi \vee \psi \notin \mathrm{Sch}(\Lambda^n) = \Lambda^\nu$.

Hence, contrapositively, if the disjunction $\varphi \vee \psi$ is in Λ^ν then so is at least one of φ and ψ. $\square$

The converse is not true, as we shall see in the next section: there are logics without disjunction property whose negative closure *does* have the disjunction property.

5.2 Stability of intermediate logics

In this section we will get more concrete and undertake an examination of the effect of the negative closure operation on some of the best-known intermediate logics. Table 5.2 introduces (or recalls) the definition of the logics we shall be

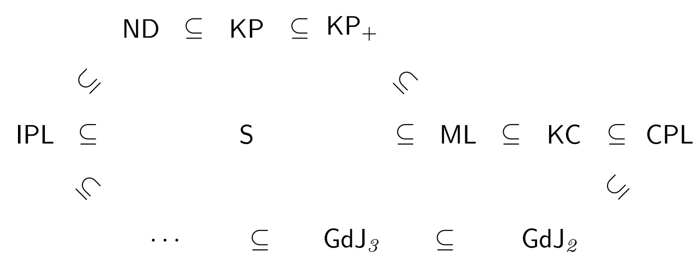


Figure 5.1: An overview of the logic discussed in this section.

concerned with in the present section.

Name	Axioms or characterization
ND	$\mathtt{ND}_k := (\neg p \to \bigvee_{i\leq k} \neg q_i) \to \bigvee_{i \leq k}(\neg p \to \neg q_i)$ for all $k \in \omega$
KP	$\mathtt{KP} := \neg p \to (q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$
S	$\mathtt{S} := ((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p$
ML	$\mathsf{Log}(\mathbf{Med})$
GdJ_k	$\mathsf{Log}(k-\text{ary trees})$
KC	$\neg p \vee \neg\neg p$
CPL	$p \vee \neg p$

One thing we can immediately see is that classical logic is stable under negative closure; this can be seen directly, because CPL *already* contains $\neg\neg p \to p$, so $\mathsf{CPL}^n = \mathsf{CPL}$, but also trivially follows from the fact that $(\ )^\nu$ is extensive on the lattice of intermediate logic and CPL is the top element of the lattice. Intuitionistic logic is stable as well, but in order to see this some work is required.

Definition 5.2.1 (Everywhere branching trees)**.** We say a tree $\mathbb{T}$ is *everywhere* branching if no point has only one immediate successor; if, in other words, any non-terminal point has at least two successors.

The following proposition states that every logic that is characterized by a class of everywhere branching trees is stable.

Proposition 5.2.2. If K is a class of finite everywhere branching trees then $\mathsf{Log}(K)$ is a stable logic.

Proof. Take a formula $\varphi = \varphi(p_1, \ldots, p_n)$ that is not in $\mathsf{Log}(K)$, we have to show that $\varphi \notin \mathsf{Log}(K)^\nu$. Now, according to proposition 3.2.11 we have $\mathsf{Log}(K)^\nu = \mathrm{Sch}(\mathsf{Log}(K)^n) = \mathrm{Sch}(\mathsf{Log}(\mathbf{n}K))$; so, it suffices to show that there is a negative model over a frame in K that falsifies the formula $\varphi(p_1 \vee q_1, \ldots, p_n \vee q_n)$.

Since $\varphi \notin \mathsf{Log}(K)$ there must be an everywhere branching tree $\mathbb{T} = (T, \leq)$ where φ is falsified, so there must be a point $w \in T$ and a valuation V on $\mathbb{T}$ with $\mathbb{T}, V, w \nVdash \varphi$.

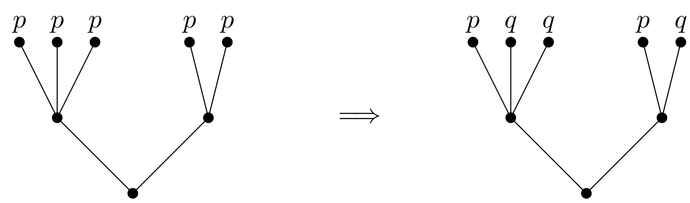


Figure 5.2: Removal of a criticality in an everywhere branching tree

We are now going to define a new valuation $\widehat{V}$ on $\mathbb{T}$, a valuation that makes $p_i \vee q_i$ true at exactly the same points where p_i is true according to V and that, moreover, is negative.

Let σ be a function that associates to each non-terminal point one of its immediate successors[1] Let $L := \sigma[T]$ be the set of points of the shape $\sigma(v)$ for some $v \in T$, and let $R := T - L$ be the set of the other points. Clearly, the sets L and R —which we think of, intuitively, as the 'left' and the 'right' successors— form a partition of T.

If $S \subseteq T$, denote by $\uparrow S$ the up-closure of S, that is, $\uparrow S := \{v \in T \mid$ there is a $u \in S$ with $u \leq v\}$. Then define the new valuation $\widehat{V}$ as follows:

- $\widehat{V}(p_i) = \uparrow(V(p_i) \cap L)$;
- $\widehat{V}(q_i) = \uparrow(V(p_i) \cap R)$.

It is clear that $\widehat{V}$ is defined in such a way as to be persistent. Moreover, $\widehat{V}$ makes $p_i \vee q_i$ true precisely where V makes p_i true, as shown by the following two inequalities:

- $\widehat{V}(p_i) \cup \widehat{V}(q_i) = \uparrow(V(p_i) \cap L) \cup \uparrow(V(p_i) \cap R) \subseteq \uparrow V(p_i) \cup \uparrow V(p_i) = \uparrow V(p_i) = V(p_i)$;
- $V(p_i) = V(p_i) \cap (L \cup R) = (V(p_i) \cap L) \cup (V(p_i) \cap R) \subseteq \uparrow(V(p_i) \cap L) \cup \uparrow(V(p_i) \cap R) = \widehat{V}(p_i) \cup \widehat{V}(q_i)$.

Thus, by lemma 3.4.18, for any point v we have $\mathbb{T}, V, v \Vdash \varphi(p_1, \ldots, p_n) \iff \mathbb{T}, \widehat{V}, v \Vdash \varphi(p_1 \vee q_1, \ldots, p_n \vee q_n)$; in particular, $\mathbb{T}, \widehat{V}, w \Vdash \varphi$.

It remains to show that $\widehat{V}$ is a negative valuation. Towards a contradiction, suppose there were points validating the double negation of an atom but not the atom itself; let v be a maximal point at which this happens.

First suppose the atom in question is one of the q_i. Consider $\sigma(v)$: $\sigma(v)$ is an *immediate* successor of v, and since our frame is a tree, no element below $\sigma(v)$ can force q_i, otherwise v should force q_i as well, while by assumption $\mathbb{T}, \widehat{V}, v \nVdash q_i$. This shows that *if* $\sigma(v) \in \widehat{V}(q_i)$, then $\sigma(v)$ must be in $V(p_i) \cap R$. But $\sigma(v) \notin R$ by definition of R, so in fact $\sigma(v) \notin \widehat{V}(q_i)$.

[1] In general, the existence of such a function is guaranteed by the axiom of choice; however, this is not really needed for the concrete cases that we shall be concerned with.

On the other hand, $\sigma(v)$ is a successor of v and $\mathbb{T}, \widehat{V}, v \Vdash \neg\neg q_i$, so by persistence also $\mathbb{T}, \widehat{V}, \sigma(v) \Vdash \neg\neg q_i$. Thus, $\sigma(v) > v$ validates the double negation of q_i but not q_i itself, contradicting the maximality of v.

If on the other hand the atom in question is one of the p_i, then consider an immediate successor $u \neq \sigma(v)$ of v: such a successor exists since $\mathbb{T}$ is everywhere branching. Repeat the same argument as above where now u plays the role of $\sigma(v)$ to show that u is a successor of v that forces $\neg\neg p_i$ but not p_i, thus contradicting the maximality of v.

This proves that $\widehat{V}$ is negative. In conclusion, we have falsified a substitution instance of φ on a negative model $(\mathbb{T}, \widehat{V})$ based on a frame $\mathbb{T} \in K$, thus showing that $\varphi \notin \mathrm{Sch}(\mathsf{Log}(\mathbf{n}K))$ □

As a corollary we immediately get the stability of several intermediate logics, in the first place of intuitionistic logic itself.

Corollary 5.2.3 (Stability of IPL)**.** $\mathsf{IPL}^{\nu} = \mathsf{IPL}$.

Proof. It is well-known that any formula φ that is not intuitionistically valid can be falsified on a finite tree: for, a finite countermodel can always be turned into a finite tree by the standard unraveling procedure. Moreover, a model M on a finite tree can always be expanded into a model M' on a finite *everywhere branching* tree such that there is a p-morphism $\eta : M' \to M$ via the following, trivial procedure: whenever in M there is a point w that has only one immediate successor, simply *duplicate* that successor together with all its generated subtree, leaving the valuation unchanged.

This shows that IPL is the logic of the class of *all* finite everywhere branching trees. An application of the previous proposition yields the stability of IPL. □

Other logics that are stable as a consequence of the previous proposition are the Gabbay-de Jongh logics.

Definition 5.2.4. For any natural k, let K_{kT} denote the class of k-ary trees, i.e., trees in which any point has at most k immediate successors. The Gabbay-de Jongh logic of index k is defined as the logic of $k-$ary trees, $\mathsf{GdJ}_k := \mathsf{Log}(K_{kT})$.

Corollary 5.2.5 (Stability of the Gabbay-de Jongh logics)**.** $(\mathsf{GdJ}_k)^{\nu} = \mathsf{GdJ}_k$ for all $k \geq 2$.

Proof. It is known (Gabbay and de Jongh, 1974, see) that if a formula is falsifiable on a k-ary tree then it is falsifiable on a *finite* k-ary tree; since $k \geq 2$, one can then turn this into an *everywhere branching* k-ary tree by the simple duplication procedure described in the proof of corollary 5.2.3. □

Another stable logic is Scott logic, axiomatized by the Scott axiom $((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p$. The proof of this fact requires some preparation.

Call F a *Scott frame* in case the frame $\mathcal{F}_S$ in figure 5.3 is not a p-morphic image of a generated subframe in F. Scott logic S is known to be sound and complete with respect to the class of finite Scott frames (see, for instance, Chagrov and Zakharyaschev, 1997, page 55 and theorem 11.58).

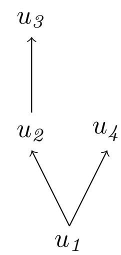


Figure 5.3: $\mathcal{F}_S$

Thus, again by 3.2.11, in order to show that the negative closure does not yield new schematic validities, it is sufficient to show that any $\varphi \notin \mathsf{S}$ has a substitution instance that can be falsified on a finite negative Scott model. In fact, just like in all cases considered so far, we will show that if $\varphi(p_1, \ldots, p_n) \notin \mathsf{S}$, then for a substitution instance that is not in S^n it is sufficient to look at the formula $\varphi(p_1 \vee q_1, \ldots, p_n \vee q_n)$ obtained by replacing any atom by a disjunction of two atoms.
Recall that the depth of a point in a finite[2] Kripke frame $F = (W, \leq)$ can be defined inductively as follows:

$$d_F(w) = \models \{d_F(v) + 1 \mid v > w\}$$

Lemma 5.2.6 (p-morphisms cannot increase the depth of points)**.** Let $\eta : F \to F'$ be a p-morphism between finite Kripke frames. For any point w in F, $d_F(w) \geq d_{F'}(\eta(w))$.

Proof. Inductively, assume that the claim holds for all proper successors of w. Towards a contradiction, suppose that $d_{F'}(\eta(w)) \geq d_F(w) + 1$: then there must be a successor t of $\eta(w)$ with $d_{F'}(t) \geq d_F(w)$.

Then by the back condition, there must be a successor v of w in F such that $\eta(v) = t$. By our inductive hypothesis on proper successors of w, $d_F(v) \geq d_{F'}(\eta(v)) = d_{F'}(t) \geq d_F(w)$.

But this is a contradiction, since v is a proper successor of w and therefore by definition of depth we have $d_F(w) > d_F(v)$. $\square$

We will use this simple fact to give an alternative, handier characterization of Scott frames.

Definition 5.2.7. We say that two endpoints e, e' in a Kripke model are *V-connected*, in symbols eVe', if they are connected by a path touching only points of depth 0 and 1; when it exists, such a path will be called a *V-path*.

It is easy to check that on a given Kripke frame, V is an equivalence relation between endpoints. If e is an endpoint, we denote by $V[e]$ the equivalence class of v, that is, the set of endpoints that are V-connected to e.

Definition 5.2.8. We call a Kripke frame V-connected in case any two endpoints are V-connected.

[2]This finiteness restriction can naturally be lifted accepting infinite ordinals as a measure of depth, but a greater generality is of no interest for the present purposes.

Obviously, a Kripke frame is V-connected in case there is only one equivalence class for the relation V.

Lemma 5.2.9. A finite Kripke frame is a Scott frame iff all of its generated subframes are V-connected.

Proof. Let $F = (W, \leq)$ be a finite frame, and suppose that not all of its generated subframes are V-connected. Take a maximal point w such that F_w is not V-connected. We will show that F_w can be p-morphically mapped onto the frame $\mathcal{F}_S$ in figure 5.3, which proves that F is not a Scott frame.

Now, w must have depth at least 2, since rooted frames of depth 0 or 1 are trivially V-connected. So we must be able to find an endpoint e of w and an intermediate point i with $w < i < e$. Since F_w is not V-connected, it must be the case that $V[e] \neq E_w$, so $E_w - V[e] \neq \emptyset$.

Moreover, for any successor $v > w$ it must either be $E_v \subseteq V[e]$ or $E_v \subseteq E_w - V[e]$. For, suppose towards a contradiction that v saw a point $f \in V[e]$ and a point $f' \in E_w - V[e]$: then in F_v there is no path connecting f and f' that touches only points of depth 0 or 1; otherwise, the same path would connect f and f' in F_s, and therefore we would have $f' \in V[e]$ contrary to assumption; but this means that F_v would not be V-connected, which contradicts the maximality of w.

We define a map $\eta : F_w \to \mathcal{F}_S$ as follows:

$$\eta(v) = \begin{cases} u_1 & \text{if } v = w \\ u_2 & \text{if } E_v \subseteq V[e] \text{ and } v \notin V[e] \\ u_3 & \text{if } v \in V[e] \\ u_4 & \text{if } E_v \subseteq E_w - V[e] \end{cases}$$

Finally, let us check that η is indeed a p-morphism from F_w onto $\mathcal{F}_S$.

1. $\eta(v)$ **is surjective**. We have $u_1 = \eta(w)$, $u_2 = \eta(i)$, $u_3 = \eta(e)$; moreover, we know that $E_w - V[e] \neq \emptyset$, so $u_4 = \eta(v)$ for some v.

2. **Forth condition**. The verification is utterly straightforward.

3. **Back condition**. We have to show that for any point v and any successor t of $\eta(v)$ in $\mathcal{F}_S$ there is a successor $u \geq v$ with $\eta(u) = t$.

   We distinguish several cases. If $\eta(v) = u_1$, then $v = w$ and thus the claim boils down to the surjectivity of η, which has been proved already. If $\eta(v)$ is an endpoint, (either u_3 or u_4) then t must equal $\eta(v)$ and the claim is trivial.

   It remains to check the case $\eta(v) = u_2$. If $t = u_2$, then v itself does the job. Otherwise, $t = u_3$; but by definition of η, if $\eta(v) = u_2$ then $E_v \subseteq V[e]$, so given an endpoint $e' > v$ we have $e' \in V[e]$ and thus $\eta(e') = u_3 = t$.

This completes the proof of the left-to-right direction of the lemma. For the converse implication, suppose $F = (W, \leq)$ is a finite Kripke frame such that all its generated subframes are V-connected. We have to show that no generated subframe of F can be p-morphically mapped onto $\mathcal{F}_S$.

In order to show this, consider a point $w \in W$ and a p-morphism η from F_w to $\mathcal{F}_S$. Consider two endpoints e, e' in F_w. Since F_w is V-connected, there must be a V-path $e = e_0 > d_1 < e_1 > \cdots > d_n < e_n = e'$.

Now, according to lemma 5.2.6, $\eta(e)$ must be an endpoint. First suppose $\eta(e) = u_3$. Then, by the forth condition, $\eta(d_1)$ must be a predecessor of u_3, and again by lemma 5.2.6 it must be a point of depth at most 1: so $\eta(d_1)$ must be either u_2 or u_3, and in both cases its unique endpoint will be u_3. But now by the forth condition, $e_1 > d_1$ will be mapped to an endpoint of $\eta(d_1)$, so it must be mapped to u_3.

Iterating this argument we conclude that all the endpoints e_i must be mapped to u_3, and in particular that $\eta(e) = \eta(e')$.

In case $\eta(e) = u_4$, we reach the same conclusion by an analogous argument showing that all the e_i's in the sequence (and indeed the d_i's as well) must then be mapped to u_3: .

So, in any case e and e' are mapped to the same endpoint. And by the arbitrariness of e and e', this means that all the endpoints of F_w are mapped to the same point.

But since $\mathcal{F}_S$ has *two* endpoints, one of these —call it u— is not the image of any endpoint of F_w. But then no point v can be mapped to u, otherwise by the finiteness of F_w we could take an endpoint $e \geq v$ and by the forth condition this should be mapped to u.

Hence, η is not surjective, and by the arbitrariness of η this shows that $\mathcal{F}_S$ is not a p-morphic image of F_w; finally, since this holds for any point w, F is a Scott frame. □

Definition 5.2.10 (Critical points)**.** A critical point for a letter p in a Kripke model M is a maximal point where $\neg\neg p \to p$ fails.

In other words, a point w of a Kripke model is critical for p in case it is not an endpoint and it does not force p while all of its proper successors do force p. The following fact is immediate from the definition.

Remark 5.2.11. If M is a finite Kripke model, M is negative iff there are no critical points.

It is then clear that given a Kripke model, we obtain a negative model by removing criticalities for each letter. But of course, for this construction to be useful for our purposes, we must perform this removal in a careful way: first, we must keep under control the satisfaction of formulas to ensure that this is not affected too badly by the changes; and in the second place, we must take care to turn Scott models into Scott models.

We will need to differentiate between the critical points of depth 1 and those of depth 2 or more. If w is a critical point for p of depth 1, we simply make a copy e' of an endpoint e, that imitates e except that we make q true instead of p, where q is a new letter. If the initial model was a Scott model, then the

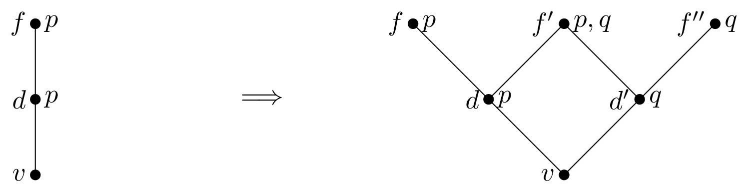


Figure 5.4: Removal of a critical point of depth ≥ 2 in a Scott frame

model which results from this operation is easily seen via lemma 5.2.9 to be a Scott model as well, because w itself V-connects the new point e' to e and thence to all the other endpoints (we will come back to the details later on).

On the other hand, if v is a critical point of depth 2 or more, we cannot simply duplicate an endpoint, since the resulting model would not be V-connected. What we need is a more complicated construction, illustrated in figure 5.4. The details of the construction are spelled out precisely by the following definition.

Definition 5.2.12. Let $M = (W, R, V)$ be a finite Kripke model.

1. Let $w_1, \ldots, w_n$ be the critical points of depth one. For $1 \leq i \leq n$, fix an endpoint e_i of w_i.
2. Let $v_1, \ldots, v_m$ be the critical points of depth two or more. For $1 \leq i \leq m$, fix points d_i and f_i such that $v_i R d_i R f_i$, d_i is an endpoint and f_i a point of depth one.

Now define the model $M^p = (W^p, R^p, V^p)$ as follows:

1. $W^p = W \cup \{e_i' \mid 1 \leq i \leq n\} \cup \{d_i', f_i', f_i'' \mid 1 \leq i \leq m\}$ where each of e_i', d_i', f_i' and f_i'' is a *new* point;
2. R^p is the reflexive transitive closure of:

   $$R \cup \{(w_i, e_i') \mid 1 \leq i \leq n\} \cup \{(d_i, f_i'), (v_i, d_i'), (d_i', f_i'), (d_i', f_i'') \mid 1 \leq i \leq m\}$$
3. V^p is defined as follows:
   - $V^p(p) = V(p) \cup \{f_i' \mid 1 \leq i \leq m\}$;
   - $V^p(q) = V(q) \cup \{e_i' \mid 1 \leq i \leq n\} \cup \{d_i', f_i', f_i'' \mid 1 \leq i \leq n\}$;
   - for any letter r different from p, q, the new points simply imitate their 'old copies', that is:

     $$V^p(r) = V(r) \cup \{e_i' \mid e_i \in V(r)\} \cup \{f_i', f_i'' \mid f_i \in V(r)\} \cup \{d_i' \mid d_i \in \widehat{V}(r)\}$$

It is clear that in the resulting model M^p all the criticalities for p have been eliminated. Also, no criticality for q has been introduced, since we only introduce successors validating q for a point v in case v already *had* a successor, and that successor does not validate q, because q is a *new* letter.

So, M^p is negative as far as p and q are concerned. Moreover, it is straightforward to check that our construction does not alter the depth of the old points: if a point u had depth d in M, then u has the same depth d in M^p. Additionally, the following lemma states that if M is a Scott model, then the model M^p resulting from this construction is a Scott model as well.

Lemma 5.2.13. For any finite Scott model M, the model M^p resulting from the construction described in definition 5.2.12 is a (finite) Scott model.

Proof. Let M be a finite Scott model. According to lemma 5.2.9, all of its generated submodels are V-connected. Since M^p is finite as well, the same lemma guarantees that our claim will be proved if we can show that also all generated submodels of the modified model M^p are V-connected.

Generated submodels of depth 0 or 1 are trivially V-connected. So, consider a point u in M^p of depth at least 2. This must be one of the old points, since points we introduce with our construction have depth 0 or 1.

Consider two endpoints e and e' of u: we have to show that e and e' are V-connected in $(M^p)_u$, that is, that in M^p there is a V-path of successors of u connecting e to e'. Here we have to make a distinction according to what e and e' are: this case checking is both extremely easy and extremely tedious. We will go through a couple of cases just to give an example of the kind of argument required and we will skip the others. The reader might be convinced by looking at figure 5.4 and noticing how we carefully create V-paths connecting the new endpoints to the old ones.

1. Case 1: both e and e' are old points. Then since F_u is V-connected, e and e' must be connected by a V-path of successors of u in M, and since our construction does not alter either the accessibility relation on the old points or the depth of the points in the frame, the same path is a V-path of successors of u in M^p.

2. Case 2: e is an old point and $e' = f_i''$ for some $1 \leq i \leq m$. By definition of the accessibility relation R^p, since u sees f_i'', u must be either d_i' or a predecessor of v_i (possibly v_i itself).

   But d_i' does not see old points, so u must be a predecessor of v_i. Then u also sees f_i, f_i', d_i and d_i'. Since F_u is V-connected, there is a V-path P of old successors of u which connects e to f_i. Then P will also be a V-path in F_u; but from f_i we can move to d_i, hence to f_i', then to d_i' and finally to f_i'', and all these points have depth 0 or 1. This shows that e can be connected to $f_i'' = e'$ by a V-path consisting of successors u in M^p.

3. Case 3: …

□

It remains to check that the modifications operated on M do not affect "too badly" the satisfaction of formulas.

Lemma 5.2.14. Let M be a finite model and let w be a point in M. For any formula φ,

$$M, w \Vdash \varphi \iff M^p, w \Vdash \varphi[^{p\vee q}/_p]$$

where $\varphi[^{p\vee q}/_p]$ denotes the formula resulting from replacing any occurrence of p with the disjunction $p \vee q$.

Proof. Define a map $\eta^p : M^p \to M$ that maps a point w in M^p to the point in M of which w is intended to be a copy:

1. $\eta^p(e_i{}') = e_i$;
2. $\eta^p(f_i{}') = \eta^p(f_i{}'') = f_i$;
3. $\eta^p(d_i{}') = d_i$;
4. if w is an old point, $\eta^p(w) = w$.

It is utterly straightforward to check that η^p is a frame p-morphism from F^p to F and that, additionally, for any point w in M^p we have:

- $M^p, w \Vdash p \vee q \iff M, \eta^p(w) \Vdash p$;
- $M^p, w \Vdash r \iff M, \eta^p(w) \Vdash r \quad$ for any $r \neq q$.

Our claim follows then immediately from lemma 3.4.18. □

Remark 5.2.15. Note that, in particular, this lemma implies that if $\neg\neg r \to r$ was true everywhere in M for $r \neq p, q$, then the same holds in M^p. This means that if M was negative for r, then so is M^p.

Equipped with these results on the properties of our construction, we are at last ready to prove the stability of Scott logic.

Theorem 5.2.16 (Stability of Scott logic)**.** $\mathsf{S}^\nu = \mathsf{S}$.

Proof. Consider a formula $\varphi(p_1, \dots, p_n) \notin \mathsf{S}$: by the aforementioned fact that S is complete with respect to finite Scott models there must be a finite Scott model M and a point w such that $M, w \nVdash \varphi(p_1, \dots, p_n)$.

Obviously, we can assume that M is only concerned with the atoms $p_1, \dots, p_n$. Now we can apply our construction for the removal of criticalities for each propositional letter occurring in φ and obtain a model $(((M^{p_1})^{p_2})\dots)^{p_n}$ with the following properties:

1. $(((M^{p_1})^{p_2})\dots)^{p_n}$ is still a Scott model. This follows from an iterated application of lemma 5.2.13;
2. $(((M^{p_1})^{p_2})\dots)^{p_n}$ it is a negative model. For, M^{p_1} is negative for p_1 and q_1 by construction; $(M^{p_1})^{p_2}$ is negative for p_2 and q_2 by construction and it is still negative for p_1 and q_1 by remark 5.2.15, and so on. In the end, $(((M^{p_1})^{p_2})\dots)^{p_n}$ will be negative for all letters, i.e. a negative model.

3. $(((M^{p_1})^{p_2})\dots)^{p_n}, w \nVdash \varphi(p_1 \vee q_1, \dots, p_n \vee q_n)$. For, lemma 5.2.14 combined with the assumption $M, w \nVdash \varphi(p_1, \dots, p_n)$ yields $M^{p_1}, w \nVdash \varphi(p_1 \vee q_1, p_2, \dots, p_n)$. Repeating the same argument for $p_2, \dots, p_n$ we finally obtain our claim.

Putting things together, $\varphi(p_1 \vee q_1, \dots, p_n \vee q_n)$ is a substitution instance of φ that can be falsified on a negative Scott model. Hence, $\varphi \notin \mathrm{Sch}(\mathsf{S}^n) = \mathsf{S}^\nu$. By the generality of φ, the equality $\mathsf{S} = \mathsf{S}^\nu$ is thus established. □

At the end of the previous section we claimed that while negative closure preserves the disjunction property (theorem 5.1.7), there are logics without disjunction property whose negative closure *does* have the disjunction property. We will now show how to use our characterization of Scott frames to obtain an example of such a logic. Recall the following result due to Jankov and de Jongh.

Theorem 5.2.17 (see Jankov (1963) and de Jongh (1968))**.** For any finite rooted frame F there exists a formula χ_F such that for any frame G we have $G \not\models \chi_F$ iff F is a p-morphic image of a generated subframe of G.

There are two possible ways to define the formulas χ_F: one is due to Jankov, the other to de Jongh; since either one will do for our purposes, we shall refer to χ_F as the *Jankov-de Jongh* formula of the frame F. Now, the idea is to use for our proof the Jankov-de Jongh formulas $\chi_{\mathcal{F}_S}$ and $\chi_{\mathcal{F}_J}$ of the frames $\mathcal{F}_S$ and $\mathcal{F}_J$ respectively, where $\mathcal{F}_J$ is the frame depicted in figure 5.5. However, we saw above that the Scott axiom S acts as a Jankov-de Jongh formula for the frame $\mathcal{F}_S$, namely, it is valid on a frame G if and only if $\mathcal{F}_S$ is not a p-morphic image of a generated subframe of G (see, for instance, Chagrov and Zakharyaschev, 1997, page 55 and theorem 11.58): therefore, we can in fact use S instead of $\chi_{\mathcal{F}_S}$.

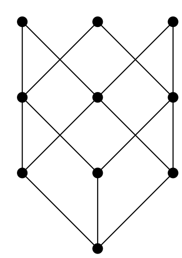


Figure 5.5: $\mathcal{F}_J$

Definition 5.2.18. Denote by KP_+ the intermediate logic obtained by expanding KP with the axiom $\mathsf{S} \vee \chi_{\mathcal{F}_J}$.

Proposition 5.2.19. KP_+ does not have the disjunction property.

Proof. By definition, $\mathsf{S} \vee \chi_{\mathcal{F}_J} \in \mathsf{KP}_+$; however, neither S nor $\chi_{\mathcal{F}_J}$ are in KP_+.

For, having $\chi_{\mathcal{F}_J} \in \mathsf{KP}_+$ would mean that $\chi_{\mathcal{F}_J}$ is derivable from $\mathsf{S} \vee \chi_{\mathcal{F}_J}$, and thus also from S, in the system KP; that is, it would amount to $\mathsf{S} \to \chi_{\mathcal{F}_J} \in \mathsf{KP}$. Analogously, having $\mathsf{S} \in \mathsf{KP}_+$ would amount to $\chi_{\mathcal{F}_J} \to \mathsf{S} \in \mathsf{KP}$. Thus, we just need to show that $\mathsf{S} \to \chi_{\mathcal{F}_J} \notin \mathsf{KP}$ and $\chi_{\mathcal{F}_J} \to \mathsf{S} \notin \mathsf{KP}$.

In order to show that a formula is not in KP, it suffices to show the existence of a frame on which each instance of the Kreisel-Putnam axiom is valid and the formula in question is not. Now, it is easy to check that any instance of the Kreisel-Putnam axiom is indeed valid on both $\mathcal{F}_S$ and $\mathcal{F}_J$ (see Chagrov and Zakharyaschev, 1997, page 55, for a characterization of the Kreisel-Putnam frames).

Obviously, $\mathcal{F}_S \not\models \mathsf{S}$ and $\mathcal{F}_J \not\models \chi_{\mathcal{F}_J}$, so it just remains to show that $\mathcal{F}_S \models \chi_{\mathcal{F}_J}$ and $\mathcal{F}_J \models \mathsf{S}$. But by the properties of Jankov formulas, this amounts to showing that $\mathcal{F}_S$ is not a p-morphic image of a generated subframe of $\mathcal{F}_J$ and viceversa.

The "viceversa" direction is immediate: $\mathcal{F}_J$ cannot be a p-morphic image of a generated subframe of $\mathcal{F}_S$, because $\mathcal{F}_J$ contains more points than $\mathcal{F}_S$.

On the other hand, observe that $\mathcal{F}_J$ is clearly V-connected, so $\mathcal{F}_J$ is a Scott frame by lemma 5.2.9 above. But by definition, being a Scott frame means precisely that $\mathcal{F}_S$ is not a p-morphic image of a generated subframe. Our claim is thus proved. □

Clearly, $\mathsf{KP} \subseteq \mathsf{KP}_+ \subseteq \mathsf{KP}{+}\mathsf{S}$. But we saw that $\mathsf{KP} \subseteq \mathsf{ML}$ (proposition 3.4.7) and $\mathsf{S} \subseteq \mathsf{ML}$ (lemma 3.4.16), whence $\mathsf{KP}_+ \subseteq \mathsf{KP}{+}\mathsf{S} \subseteq \mathsf{ML}$. Therefore, $(\mathsf{KP}_+)^\nu = \mathsf{ML}$.

This shows that KP_+ is a logic without disjunction property whose negative closure *does* have the disjunction property.

So far, the only non-stable logics we encountered are logics Λ with $\mathsf{ND} \subseteq \Lambda \subset \mathsf{ML}$, whose negative closure is Medvedev logic. However, there is another class of intermediate logics which are easily seen to be non-stable, and whose closure is classical logic.

Lemma 5.2.20. Let L be a weak intermediate logic. If $p \vee \neg p \in L$, then $L = \mathsf{CPL}$.

Proof. A straightforward inductive argument suffices to show that $\varphi \vee \neg\varphi \in L$ for any formula φ. □

As an immediate corollary of this fact, the negative closure of the logic KC of the weak excluded middle, axiomatized by $\neg p \vee \neg\neg p$, coincides with classical logic.

Corollary 5.2.21. $\mathsf{KC}^\nu = \mathsf{CPL}$

Proof. Since $\neg p \vee \neg\neg p \in \mathsf{KC}$ we have $p \vee \neg p \in \mathsf{KC}^n$ by definition of negative variant. So, by the previous proposition $\mathsf{KC}^n = \mathsf{CPL}$, whence $\mathsf{KC}^\nu = \mathrm{Sch}(\mathsf{KC}^n) = \mathrm{Sch}(\mathsf{CPL}) = \mathsf{CPL}$. □

Then, by the monotonicity of the negative closure operator, classical logic is the negative closure of any intermediate logic $\Lambda \supseteq \mathsf{KC}$, such as the Gödel-Dummett logic axiomatized by $(p \to q) \vee (q \to p)$. The following proposition shows that including KC is not only a sufficient, but also a necessary condition for the equality $\Lambda^\nu = \mathsf{CPL}$ to hold; in other words, KC is the *smallest* intermediate logic whose closure is CPL.

Proposition 5.2.22 (Characterization of the logics whose negative closure is CPL)**.** For any intermediate logic Λ, $\Lambda^\nu = \mathsf{CPL} \iff \Lambda \supseteq \mathsf{KC}$.

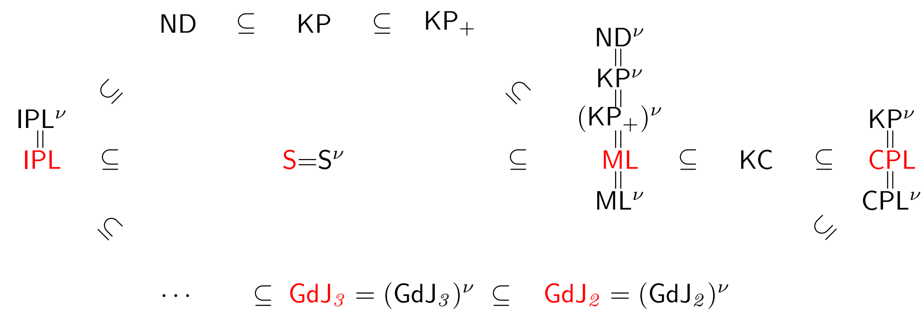


Figure 5.6: Negative closure of intermediate logics. In red the stable logics.

Proof. We have already seen that the right-to-left direction holds. For the converse implication, fix Λ and reason contrapositively: if $\mathsf{KC} \not\subseteq \Lambda$, then $\neg p \vee \neg\neg p \notin \Lambda$. But by definition of Λ^n this means that $p \vee \neg p \notin \Lambda^n$ and therefore also $p \vee \neg p \notin \Lambda^\nu$, whence $\Lambda^\nu \neq \mathsf{CPL}$. $\square$

Our findings are summarized by picture 5.2.

Chapter 6

First-order inquisitive semantics

In this chapter we leave the propositional setting and set out to develop inquisitive logic for a first-order language. This will turn out to be a problematic task, since in the first-order case the definition of possibilities as maximal supporting states completely fails to yield an adequate notion of meaning.

An analysis of the causes of this problem will lead us to revise our propositional semantics and to start over from a different definition of possibilities which, fortunately, will turn out to be tightly related to the old one, so that most features of the system we discussed so far will be preserved and most results —in particular, the logical ones— will carry over to the modified setup.

We will then show that the new definitions extend in a natural way to the first-order case, allowing for the development of a first-order inquisitive semantics, and we will conclude with some remarks about first-order inquisitive logic.

If our exposition in this chapter becomes somehow more drafty than it should, it is because the ideas that constitute the subject matter of the present chapter have been developed only very recently and are still largely in the process of evolving towards a satisfactory presentation.

6.1 First-order inquisitive semantics and the maximality problem

In this section we consider the notion of support for a first-order language $\mathcal{L}$ and discuss the problems that arise if we try to implement an analogue of the semantics discussed in chapter 2 based on this notion. We denote by $\mathcal{L}_F$ the set of function symbols in $\mathcal{L}$, and by $\mathcal{L}_P$ the set of predicate symbols. Obviously, we assume a countably infinite set of variables, that we will usually denote by $x, y, z, \ldots$.

We take as primitive the connectives $\bot, \wedge, \vee, \rightarrow$ and both quantifiers $\exists$ and

$\forall$. Besides the abbreviations we already had in the propositional case ($\neg\varphi$ for $\varphi \to \bot$, $?\varphi$ for $\varphi \vee \neg\varphi$ and $!\varphi$ for $\neg\neg\varphi$) we introduce two new ones, whose role will become clear in the course of the chapter: we shall write $\Diamond\varphi$ for $\varphi \vee \top$ and $(?x_1, \dots, x_n)\varphi$ for $\forall x_1 \dots \forall x_n ?\varphi$.

For the sake of simplicity (things will prove to be problematic enough even so) we will consider models over a fixed domain. Interpreting states as information states, this amounts to the assumption that there is no uncertainty as regards the set of individuals that are object of the discourse.

We also assume a fixed interpretation for function symbols (including constants). This amounts to the assumption that it is known who the names used in the conversation refer to, as well as all values of all functional nouns.

Fix a domain D and an interpretation f^D on D of each function symbol f in $\mathcal{L}$. We denote by $\mathbb{D}$ the structure $(D, \langle f^D \mid f \in \mathcal{L}_F \rangle)$.

Definition 6.1.1 (Models)**.** A $\mathbb{D}$-model is simply a model M for the language $\mathcal{L}$ that is based on $\mathbb{D}$, that is, a model whose reduct to the language $\mathcal{L}_F$ is $\mathbb{D}$. We denote by $\mathcal{I}_{\mathbb{D}}$ the set of all $\mathbb{D}$-models.

Definition 6.1.2 (States)**.** A $\mathbb{D}$-*state* is a set of $\mathbb{D}$-models.

We will use s, t, u as meta-variables for states. Reference to $\mathbb{D}$ will be dropped whenever possible.

Recall that an *assignment* into D is simply a map $g : \mathtt{Var} \to D$. If g is an assignment into D, $x \in \mathtt{Var}$ and $d \in D$, we denote by $g[x \mapsto d]$ the assignment mapping x to d and otherwise behaving like g.

Definition 6.1.3 (Classical meaning)**.** Given a formula φ and a model M, we write $M, g \models \varphi$ in case φ is true classically in M under the assignment g.

Given a formula φ, we denote by $|\varphi|_g$ the set of all $\mathbb{D}$-models in which φ is classically true under the assignment g. If $\varphi = \varphi(x_1, \dots, x_n)$ is a formula whose free variables are among $x_1, \dots, x_n$, then obviously all that matters about the assignment g is the value it assigns to those variables. So, if $d_1, \dots, d_n$ are elements of D, we will also write $M \models \varphi(d_1, \dots, d_n)$ and $|\varphi(d_1, \dots, d_n)|$ to mean, respectively, $M, g \models \varphi$ and $|\varphi|_g$ where g is an assignment mapping each x_i to d_i.

The definition of support for first-order formulas is a very natural generalization of its propositional counterpart.

Definition 6.1.4 (Support)**.** Let s be a state and g a valuation.

1. $s, g \models \varphi$ iff $s \in |\varphi|_g$ for φ atomic
2. $s, g \models \bot$ iff $s = \emptyset$
3. $s, g \models \varphi \wedge \psi$ iff $s, g \models \varphi$ and $s, g \models \psi$
4. $s, g \models \varphi \vee \psi$ iff $s, g \models \varphi$ or $s, g \models \psi$
5. $s, g \models \varphi \to \psi$ iff $\forall t \subseteq s$, if $t, g \models \varphi$ then $t, g \models \psi$

6. $s, g \models \forall x\varphi$ iff for all $d \in D$, $s, g[x \mapsto d] \models \varphi$
7. $s, g \models \exists x\varphi$ iff there is a $d \in D$ such that $s, g[x \mapsto d] \models \varphi$

Notation. Again, if $\varphi(x_1, \ldots, x_n)$ is a formula whose variables are among $x_1, \ldots, x_n$, then obviously whether or not $s, g \models \varphi$ holds depends only on the behaviour of g on the variables $x_1, \ldots, x_n$, so if $d_1, \ldots, d_n$ are elements of D, we also write $s \models \varphi(d_1, \ldots, d_n)$ to mean $s, g \models \varphi$ where g is any assignment such that $g(x_i) = d_i$ for $1 \leq i \leq n$.

In particular, if φ is a sentence we will simply omit any reference to the assignment and write $s \models \varphi$.

Just like in the propositional case, the atomic clause simply requires φ to be true in all models in the given state. Also, like in the propositional case the empty state is inconsistent and easy inductive arguments suffice to prove the following two facts.

Proposition 6.1.5 (Persistence)**.** If $t \subseteq s$ and $s \models \varphi$, then $t \models \varphi$.

Proposition 6.1.6 (Classical behaviour of singletons)**.** For any model M, any assignment g and formula φ,

$$\{M\}, g \models \varphi \iff M, g \models \varphi$$

Something well-known from chapter 2 that shows up again and will soon turn out useful is the classical behaviour of negations. For, spelling out the definition of support for negation, we have the following fact.

Remark 6.1.7. For any $\mathbb{D}$-state s, any assignment g and any formula φ,

$$s, g \models \neg\varphi \iff M, g \models \neg\varphi \text{ for all } M \in s$$

In particular, a declarative $!\varphi$ is supported by a state exactly in case all models in the states make φ true.

The maximality problem. Based on the notion of support, we can extend the propositional definition of informativeness and inquisitiveness to the first-order case, prove equivalent characterizations of these notions and do other such things.

But there is something that is crucial to the inquisitive programme that we *cannot* do: we cannot associate to a formula a *proposal*, a set of possible updates, the equivalent of the inquisitive meaning we had in chapter 2. The reason is that in the first-order case, the maximalization involved in the definition of possibilities is extremely problematic.

More precisely, the point is that the equivalent of proposition 2.1.10 fails: a state supporting a formula may not be included in a maximal one; indeed, there are even (meaningful) formulas that have *no* maximal supporting states, as we are now going to show.

Proposition 6.1.8. There is a formula φ that has no maximal supporting states.

Proof. Let our language consist of a binary function symbol $+$ and a unary predicate symbol P; let our domain be the set ω of natural numbers and let $+$ be interpreted as addition. Moreover, let $x \leq y$ abbreviate $\exists z(x + z = y)$.

Consider the formula $\exists x B(x)$, where $B(x)$ denotes the formula $!\forall y(P(y) \to y \leq x)$. We call $\exists x B(x)$ the *boundedness* formula. By remark 6.1.7, a state s supports $B(n)$ for a certain number n if and only if $B(n)$ is true in all models in s; and clearly, $B(n)$ is true iff the number n is an upper bound for the set of numbers with the property P: for any first-order model M, $M \models B(n)$ iff $P^M \subseteq \{0, \ldots, n\}$, where P^M denotes the extension of the predicate P in M.

We will show that any state that supports $\exists x B(x)$ can be extended to a bigger state that still supports the same formula. Consider an arbitrary state s supporting $\exists x B(x)$. This means that there is a natural n such that $s \models B(n)$; so $B(n)$ must be true in any model $M \in s$, which means that for any such M, $P^M \subseteq \{0, \ldots, n\}$.

Now let M^* be the model defined by $P^{M^*} = \{n+1\}$. $M^* \notin s$ since we just saw that the extension of P in any model in s is bounded by n; hence $s \cup \{M^*\}$ is a proper superset of s. It is obvious that for any model $M \in s \cup \{M^*\}$ we have $P^M \subseteq \{0, \ldots, n+1\}$ and thus $M \models B(n+1)$. Hence, according to our earlier remark about the behaviour of $B(x)$ we have $s \cup \{M^*\} \models B(n+1)$ and therefore $s \cup \{M^*\} \models \exists x B(x)$. So, $s \cup \{M^*\}$ is a proper extension of s that still supports $\exists x B(x)$.

By the arbitrariness of s, this argument shows that no state supporting the boundedness formula can be maximal. ☐

In order to gather some clues as to where to head in order to overcome this difficulty, let us meditate shortly on this example. What possibilities did *we* expect to come out of the boundedness example? Now, $B(x)$ is simply supported whenever it is true, so it has a classical behaviour. The existential quantifier in front of it, on the other hand, is designed to be satisfied only by the knowledge of a concrete bound, just like in the propositional case a disjunction (of assertions) is designed to only be satisfied by the knowledge of a disjunct.

Therefore, what we would expect from the boundedness formula is a hybrid behaviour: of course, it should inform that there is an upper bound of P; but it should also raise the issue of *what* number is an upper bound of P. The possible resolutions of this issues are $B(0), B(1), B(2)$,etc., so the possibilities for the formula should be $|B(0)|, |B(1)|, |B(2)|$, etc.

Now, the definition of possibilities through maximalization has the effect of selecting *alternative* ways to resolve the issue raised by a formula. The problem is that obviously, if 0 is a bound of P, then so are $1, 2$, etc.; if 1 is a bound, then so are $2, 3$, etc. So, the ways in which the issue raised by the boundedness formula can be resolved cannot be regarded as *alternatives*. Still, $B(0), B(1)$, etc. are genuine solutions to the meaningful issue raised by the existential, and our semantics should be able to model this.

This indicates that we need to come up with another way of associating a proposal to a formula; and if we are to be able to deal with the boundedness example, we need our notion to encompass proposals containing non-alternative possibilities.

Notice that we cannot hope for a definition of such possibilities in terms of support. For, consider the following variant of the boundedness formula: $\exists x(x \neq 0 \wedge B(x))$. Possibilities for this formula should correspond to the possible witnesses for the existential, and since 0 is *not* a witness, we expect $|B(0)|$ *not* to be a possibility.

Thus, a system that represents the inquisitive behaviour of the existential quantifier in a satisfactory way should associate different possibilities to the formulas $\exists x B(x)$ and $\exists x(x \neq 0 \wedge B(x))$. However, as far as support is concerned, the two formulas are equivalent. This is because, as soon as one knows an upper bound for the set P, one immediately knows a *positive* upper bound, so the two formulas are resolved in exactly the same information states.

The notion of support describes the knowledge conditions under which a formula is resolved, but it is not sufficiently fine-grained to determine the what its *ways of being resolved* are. We shall return to this point more extensively later on, showing that indeed, support constitutes the 'extensional shadow' of possibilities.

This discussion indicates that we need to devise a direct, support-independent notion of possibilities. In the next section we turn to this task, starting over with a new approach right from the propositional case. However, we have not worked our way through the previous five chapters in vain: we will see that it is possible to deal with non-alternative possibilities while at the same time preserving most of the essential features and results discussed in the previous sections.

6.2 Propositional possibility semantics

6.2.1 Propositions

In this section we are going to redefine possibilities and to devise a new notion of inquisitive meaning that encompasses the old one. If we are to define possibilities avoiding recourse to maximalization, the obvious idea is to give an inductive definition. Looking back at the definition of support, the natural candidate is clearly the following (cf. the proof of proposition 2.1.13).

Definition 6.2.1 (Propositions)**.**

1. $[\![p]\!] = \{|p|\}$ if $p \in \mathcal{P}$
2. $[\![\bot]\!] = \{\emptyset\}$
3. $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]$
4. $[\![\varphi \wedge \psi]\!] = \{s \cap t \mid s \in [\![\varphi]\!] \text{ and } t \in [\![\psi]\!]\}$

5. $[\![\varphi \to \psi]\!] = \{\Pi_f \mid f : [\![\varphi]\!] \to [\![\psi]\!]\}$,

$$\text{where } \Pi_f = \{w \in I \mid \text{ for all } s \in [\![\varphi]\!], \text{ if } w \in s \text{ then } w \in f(s)\}$$

We refer to $[\![\varphi]\!]$ as the *proposition expressed by* φ instead of the *meaning* of φ in order to avoid confusion with the notion $[\varphi]$ we had in chapter 2. The *possibilities* for φ are simply the elements of the proposition $[\![\varphi]\!]$.

A first, important feature of the notion of proposition is that it relates to support in the natural way. The straightforward inductive proof of this fact is omitted.

Proposition 6.2.2. For any state s and formula φ,

$$s \models \varphi \iff s \subseteq t \text{ for some possibility } t \in [\![\varphi]\!]$$

The next proposition shows that as far as *maximal* possibilities —that is, *alternatives*— are concerned, the new semantics coincides with the old one. In particular, all the possibilities we had in chapter 2 are still possibilities under the new definitions: only, we may now have additional possibilities that are included in other possibilities.

Proposition 6.2.3. For any state s and formula φ,

$$s \in [\varphi] \iff s \text{ is a maximal element of } [\![\varphi]\!]$$

Proof. If $s \in [\varphi]$, then $s \models \varphi$ and so, by proposition 6.2.2, $s \subseteq t$ for some $t \in [\![\varphi]\!]$; then again by the same proposition, $t \models \varphi$, whence $s = t$ by the maximality of s.

On the other hand, if s is a maximal element of $[\![\varphi]\!]$, then again by 6.2.2 we have $s \models \varphi$; now suppose $s \subseteq t \models \varphi$: then t must be included in a $u \in [\![\varphi]\!]$, and by the maximality of s we have $s = u \supseteq t \supseteq s$ and therefore $s = t$. This shows that s must be a maximal state supporting φ, whence $s \in [\varphi]$. □

We will see that, in the propositional case, maximal possibilities embody the *issue* raised by a formula. Hence, the previous proposition shows that possibility semantics does not depart from the maximalization semantics in the treatment of issues.

Moreover, the following corollary shows that, as expected, the new semantics also does not depart from the classical treatment of information. For, as explained in chapter 2, the union $\bigcup[\![\varphi]\!]$ of the possibilities for φ expresses the informative content of φ, as the acceptance of φ implies the elimination of those indices that are not contained in any possibility for φ.

Corollary 6.2.4. $\bigcup[\![\varphi]\!] = |\varphi|$.

Proof. It follows from proposition 6.2.3 that any possibility $s \in [\![\varphi]\!]$ is included in a maximal one. Thus, $\bigcup[\![\varphi]\!] = \bigcup[\varphi] = |\varphi|$ where the last equality is given by proposition 2.1.11. □

Another thing that is immediately seen from the inductive definitions of possibilities is that Groenendijk's inequalities carry over the new setting (in particular, any formula will still have only a finite number of possibilities). Denoting by $\#X$ the cardinality of a set X, we have:

Remark 6.2.5 (Groenendijk's inequalities)**.**

1. $\#[\![p]\!] = \#[\![\bot]\!] = 1$
2. $\#[\![\varphi \vee \psi]\!] \leq \#[\![\varphi]\!] + \#[\![\psi]\!]$
3. $\#[\![\varphi \wedge \psi]\!] \leq \#[\![\varphi]\!]\#[\![\psi]\!]$
4. $\#[\![\varphi \to \psi]\!] \leq \#[\![\psi]\!]^{\#[\![\psi]\!]}$

Call a formula φ an *assertion* in case its proposition consists of only one possibility (we will later give an "official", but equivalent, definition of assertions). By the equality $\bigcup[\![\varphi]\!] = |\varphi|$, this unique possibility must then be $|\varphi|$, so assertions essentially behave classically, in the sense that they simply propose to establish the corresponding fact.

The previous inequalities immediately entail a few sufficient syntactic conditions on a formula to be an assertion, which the reader will find familiar from chapter 2.

Corollary 6.2.6. For any propositional letter p and any formulas φ and ψ,

1. p is an assertion;
2. $\bot$ is an assertion;
3. if both φ and ψ are assertions, then so is $\varphi \wedge \psi$;
4. if ψ is an assertion, then so is $\varphi \to \psi$.

In particular —just like in chapter 2— disjunction-free formulas and negations are assertions. Thus, disjunction is the unique source of non-classical behaviour in the semantics.

Corollary 6.2.7. Disjunction-free formulas are assertions.

Corollary 6.2.8. Any negation $\neg\varphi$ is an assertion.

6.2.2 Resolutions

In order to gather some intuition about the new notion of possibilities, let us go back to the interpretation of support as 'knowing how' discussed in section 2.4. There we claimed that the possibilities for a formula φ *mirror* the alternative ways in which a formula may be realized; however, we have not looked at *this* side of the mirror, in the sense that we have not specified what these 'ways of being realized' are. The time has come to formalize that intuition and capitalize on it.

Interpreting the definition of support, we said that atomic formulas can be realized in only one way, namely by being true. We said that in order to know how $\varphi \vee \psi$ is realized it suffices to know how either disjunct is realized, so that a way of realizing the disjunction is a way of realizing either disjunct. We said that knowing how $\varphi \wedge \psi$ is realized amounts to knowing how each conjunct is realized, so that a way of realizing the conjunction consists of a way of realizing the first conjunct and a way of realizing the second. Finally, we said that knowing how an implication is realized means knowing a function that turns information as to how the antecedent is realized into information as to how the consequent is.

This suggests an inductive definition of the set of ways in which a formula φ is realized. In order to avoid multiple copies of the same realization, we need to adopt a little device. Let us choose a normal form for formulas *in classical logic* such that the normal form of each formula contains only negations and conjunctions, and denote by φ_{nf} the normal form of a formula φ; we may assume that $\bot_{nf} = \bot$ and that $p_{nf} = p$ for propositional letters. We can then define *realizations* as follows.

Definition 6.2.9 (Realizations/resolutions)**.**

1. $\mathcal{R}(p) = \{p\}$ for $p \in \mathcal{P}$
2. $\mathcal{R}(\bot) = \{\bot\}$
3. $\mathcal{R}(\varphi \vee \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$
4. $\mathcal{R}(\varphi \wedge \psi) = \{(\rho \wedge \sigma)_{nf} \mid \rho \in \mathcal{R}(\varphi) \text{ and } \sigma \in \mathcal{R}(\psi)\}$
5. $\mathcal{R}(\varphi \to \psi) = \{(\bigwedge_{\rho \in \mathcal{R}(\varphi)} (\rho \to f(\rho)))_{nf} \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$

It is immediate to see inductively that $\mathcal{R}(\varphi)$ is finite for all formulas, and this also insures that the last clause in the definition is well-formulated. Incidentally, observe that the last clause can also be formulated as follows.

5b. $\mathcal{R}(\varphi \to \psi) = \{(\bigwedge_{1 \leq i \leq n} (\rho_i \to \sigma_i))_{nf}$

where $\{\rho_1, \ldots, \rho_n\} = \mathcal{R}(\varphi)$ and $\{\sigma_1 \ldots, \sigma_n\} \subseteq \mathcal{R}(\psi)\}$

Terminology. Recall that the interpretation of support as 'knowing how' is related to its interpretation in terms of proposals by the last remark of section 2.4: the effect of a formula φ can be described as informing *that* φ is true and raising the issue about *how* φ is realized; in this perspective, the realizations of φ are (up to equivalence) precisely the formulas that resolve the issue raised by φ. Therefore, we will also —and in fact mainly— refer to the elements of $\mathcal{R}(\varphi)$ as the *resolutions* of φ; this terminology will help make many results in this section more intuitively clear.

We said earlier that possibilities mirror the different ways a formula can be realized, or resolved. This should be understood literally.

Remark 6.2.10 (Possibilities mirror resolutions)**.** For any formula φ, the truth-set map from $\mathcal{R}(\varphi)$ to $[\![\varphi]\!]$ defined by $\rho \mapsto |\rho|$ is a one-to-one correspondence; in particular,

$$[\![\varphi]\!] = \{|\rho| \mid \rho \in \mathcal{R}(\varphi)\}$$

Proof. The injectivity of the map is guaranteed by the fact that if two resolutions have the same classical meaning, then they have the same normal form and therefore they coincide, since resolutions are in normal form by definition. In view of this remark, the equality $[\![\varphi]\!] = \{|\rho| \mid \rho \in \mathcal{R}(\varphi)\}$ is then immediately clear once one compares the inductive clauses in the definition of possibilities with those in the definition of resolutions. □

Thus, resolutions provide the syntactic counterpart of possibilities, and we can work with either notion depending on which approach is more convenient for the purpose at hand. This correspondence also provides a formal ground to our claim in section 4.3 that the elements of $[\varphi]$, which as we saw are nothing but maximal elements of $[\![\varphi]\!]$, mirror the *alternative* ways of resolving the formula φ. In terms of resolutions, proposition 6.2.2 can be restated as follows.

Corollary 6.2.11. For any state s and any formula φ,

$$s \models \varphi \iff s \subseteq |\rho| \text{ for some } \rho \in \mathcal{R}(\varphi)$$

This can be read as saying that a formula is supported in a state in case it is *resolved*, that is, if a resolution of it is known in the state.

Observe that by definition, resolutions are disjunction-free: thus, according to corollary 6.2.7, $[\![\rho]\!] = \{|\rho|\}$ for any resolution ρ of a formula φ; this shows that, as expected, resolutions are assertions, and do not raise further issues.

6.2.3 Strong entailment

It is clear that, while support and maximal possibilities are interdefinable via the 'support iff included in a possibility' connection, support and maximal possibilities do not completely determine the proposition expressed by a formula. For instance, both $\top$ and $\top \vee p$ are supported by all states, but $[\![\top]\!] = \{\mathcal{I}\}$ while $[\![\top \vee p]\!] = \{\mathcal{I}, |p|\}$. Thus, it will be convenient to have, alongside the usual notions of entailment and equivalence in terms of support, a finer variant of them which is sensible to those differences in meaning that are not detected by support.

Definition 6.2.12. Given formulas φ and ψ, we say that φ *strongly entails* ψ, in symbols $\varphi \mid\!\mid\!\!\sim \psi$, in case $[\![\varphi]\!] \subseteq [\![\psi]\!]$. In case $\varphi \mid\!\mid\!\!\sim \psi$ and $\psi \mid\!\mid\!\!\sim \varphi$ we write $\varphi \sim \psi$ and say that φ and ψ are *strongly equivalent*.

Thus, $\varphi \sim \psi$ in case $[\![\varphi]\!] = [\![\psi]\!]$. As expected, strong entailment implies entailment (and thus, strong equivalence implies equivalence).

Remark 6.2.13. If $\varphi \mid\!\mid\!\!\sim \psi$ then $\varphi \models \psi$.

Proof. If $[\![\varphi]\!] \subseteq [\![\psi]\!]$ then by 6.2.2 any state that supports φ is included in a possibility $s \in [\![\varphi]\!] \subseteq [\![\psi]\!]$ and therefore by persistence it supports ψ. □

The difference between entailment and strong entailment can be understood as follows. Recalling that a state supports a formula in case the formula is resolved in that state, '$\varphi \models \psi$' means that ψ is resolved whenever φ is, so that once we resolve φ we will be able to resolve ψ as well. Hence, entailment is a notion of relative resolubility.

On the other hand, $\varphi \mid\!\mid\!\sim \psi$ means that whenever φ is resolved, ψ is resolved *in the same way*: any resolution of φ also resolves ψ. In other words, plain entailment is in a sense "extensional" in that it only takes into account *when* a formula is resolved, while strong entailment is "intensional", as it takes into account not only *when*, but also *how* a formula is resolved.

While in the propositional case equivalence boils down to having the same maximal possibilities, we shall see that things become quite interesting in the first order case, where we will give two formulas that have no common resolution/possibility —that is, the responses that they invite are totally different— and yet they are equivalent, since a solution to the one can always be derived from a solution to the other.

Clearly, both entailment notions are meaningful and interesting from the inquisitive perspective. The formal properties of inquisitive entailment have been studied in depth in chapter 3; a syntactic characterization of strong entailment will be discussed in section 6.2.7.

6.2.4 Disjunctive normal form and expressive completeness

The central idea of inquisitive semantics is that formulas propose a choice between one or more updates. We saw that these updates are named by the resolutions of the formula. Moreover, by definition of the semantics, a disjunction has the effect of introducing choices. Therefore, it should hardly be surprising that any formula is strongly equivalent to the disjunction of its resolutions.

Proposition 6.2.14. For any formula φ, $\varphi \sim \bigvee \mathcal{R}(\varphi)$.

Proof. Fix φ. We saw that $[\![\rho]\!] = \{|\rho|\}$ for any resolution $\rho \in \mathcal{R}(\varphi)$, so by the definition of possibilities for disjunction, $[\![\bigvee \mathcal{R}(\varphi)]\!] = \bigcup_{\rho \in \mathcal{R}(\varphi)} [\![\rho]\!] = \bigcup_{\rho \in \mathcal{R}(\varphi)} \{|\rho|\} = \{|\rho| \mid \rho \in \mathcal{R}(\varphi)\}$.

On the other hand by remark 6.2.10 also $[\![\varphi]\!] = \{|\rho| \mid \rho \in \mathcal{R}(\varphi)\}$. Thus, $\varphi \sim \bigvee \mathcal{R}(\varphi)$. □

Since the disjuncts in $\bigvee \mathcal{R}(\varphi)$ are uniquely determined by the proposition $[\![\varphi]\!]$, this result gives a disjunctive normal form representation for formulas in possibility semantics.

Here we are *extremely* close to the disjunctive negative translation DNT discussed in chapter 3. For, it is also easy to check that $\varphi \sim \textsc{dnt}(\varphi)$; in fact, DNT($\varphi$) is nothing but $\bigvee \mathcal{R}(\varphi)$ "in disguise", where each disjunct is manipulated into the shape of an equivalent negation.

Whether we use DNT or we simply note that resolutions are assertions and thus equivalent to their double negation by corollary 6.2.8, we can draw the following conclusion.

Corollary 6.2.15. Any formula is equivalent to a disjunction of negations.

In particular, any formula is essentially a disjunction of assertions. Moreover, since negations behave classically, within the scope of negations we can safely substitute classically equivalent formulas. And since the set of connectives $\{\neg, \vee\}$ is complete in classical logic, we can replace each disjunct in $\bigvee_{\rho \in \mathcal{R}(\varphi)} !\rho$ with an equivalent one in which only negation and disjunction occur. Thus, any formula is equivalent to one containing only negations and disjunctions.

As predictable, something slightly stronger can be said if we proceed bottom-up instead of top-down, namely that any set of states is also the meaning of some formula, which can be constructed using negation and disjunction only.

Proposition 6.2.16 (Expressive completeness of $\{\neg, \vee\}$)**.** If $\mathcal{P}$ is a finite set of propositional letters, for any set Π of $\mathcal{P}-$states there is a formula φ_Π containing only negation and disjunction with $[\![\varphi_\Pi]\!] = \Pi$.

Proof. For any state $s \in \Pi$, let χ_s be a formula containing only negations and disjunctions with $|\chi_s| = s$; by proposition 6.2.8 we have $[\![!\chi_s]\!] = \{s\}$. Then $\bigvee_{s \in \Pi} !\chi_s$ is a formula containing only negations and disjunctions and $[\![\bigvee_{s \in \Pi} !\chi_s]\!] = \bigcup_{s \in \Pi} [\![!\chi_s]\!] = \bigcup_{s \in \Pi} \{s\} = \Pi$. □

It is also immediate to see that, just like in the maximalization semantics, the set of connectives $\{\neg, \wedge\}$ is "sound and complete" for the classical meanings, in the sense that a meaning can be expressed by a formula containing only negations and conjunctions if and only if it consists of one sole possibility.

6.2.5 Inquisitiveness, informativeness, suggestiveness

In section 2.1.3 we saw how inquisitive semantics can be viewed as a system in which the meaning of formulas consists of a purely informative component and a purely inquisitive component which, together, fully exhaust the meaning itself. In the present system, these two components can still be identified and in fact will behave exactly like in the old system, but they will not fully determine the proposal associated to a formula: a third dimension of meaning can be identified, consisting in the potential to *suggest* certain updates without supplying information on the matter and without having such updates as integral part of an issue. Together, these three dimensions fully exhaust the meaning of the formula.

It will be useful to first introduce the usual notions of *tautologies* —formulas that make a trivial proposition— and *contradictions* —formulas that make an unacceptable proposition.

Definition 6.2.17 (Tautologies, contradictions)**.**

1. A formula φ is a *tautology* in case $[\![\varphi]\!] = \{\mathcal{I}\}$;
2. A formula φ is a *contradiction* in case $[\![\varphi]\!] = \{\emptyset\}$.

The equality $\bigcup[\![\varphi]\!] = |\varphi|$ immediately entails that contradictions are precisely the classical contradictions; on the other hand, just like in the maximalization case, classical tautologies may well make meaningful propositions.

We start our analysis of the properties of formulas from the most standard notion, that of informativeness. We saw above that the treatment of information amounts to the classical one, so informativeness will be defined as usual: a formula is informative if it eliminates possible worlds, i.e. if it is not a classical tautology.

Definition 6.2.18 (Informativeness)**.** A formula φ is *informative* in case $|\varphi| \neq \mathcal{I}$.

Since the meaning of the negation of a formula consists in the negation of the informative content, a formula is informative if and only if its negation is not a contradiction.

Now let us turn to inquisitiveness. Intuitively, a formula φ should be inquisitive if it *requires* information from the other participants, that is, if it can only be resolved by providing additional information (additional with respect to the one φ itself supplies). Hence, we require that all resolutions ρ of φ be strictly more informative than φ, that is, $|\rho| \subsetneq |\varphi|$. But since all resolutions are at least as informative as φ, this can be simplified as follows.

Definition 6.2.19 (Inquisitiveness)**.** A formula φ is *inquisitive* in case $|\varphi| \notin [\![\varphi]\!]$.

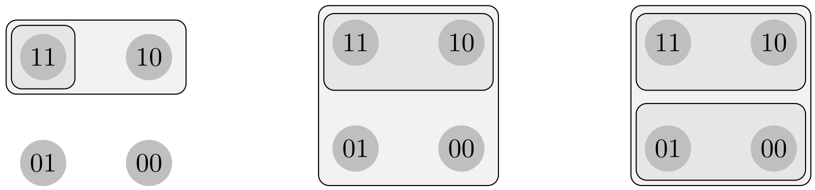
Proposition 6.2.20 (Alternative characterizations of inquisitiveness)**.** For any formula φ, the following are equivalent.

1. φ is not inquisitive;
2. φ is classically equivalent to one of its resolutions;
3. $[\![\varphi]\!]$ has a greatest element;
4. $|\varphi| \models \varphi$;
5. $\varphi \equiv {!\varphi}$.

Proof. ($1 \Leftrightarrow 2$) Follows from the definition of inquisitiveness and proposition 6.2.10.

($1 \Leftrightarrow 3$) Follows from the equality $\bigcup[\![\varphi]\!] = |\varphi|$ (proposition 6.2.4).

($1 \Leftrightarrow 4$) The implication from (1) to (4) is immediate by proposition 6.2.2. For the converse, suppose $|\varphi| \models \varphi$: by the same corollary, $|\varphi|$ must be included in a possibility $t \in [\![\varphi]\!]$; but since in turn possibilities for φ are included in $|\varphi|$ (as follows from 6.2.4) it must be $|\varphi| = t \in [\![\varphi]\!]$;



(a) $p \vee (p \wedge q)$ (b) $\Diamond p = p \vee \top$ (c) $\Diamond p \vee \Diamond \neg p$

Figure 6.1: Three examples of suggestive formulas.

($4 \Leftrightarrow 5$) By proposition 2.1.25.

□

These alternative characterizations suggest further intuitions about the notion of inquisitiveness. For instance, item (2) states that a formula is *not* inquisitive in case it resolves itself, and item (3) shows that φ is not inquisitive in case it offers an obvious, safe choice that can always be chosen by the other participants regardless of their information state.

Note that according to proposition 2.1.25, the notion we defined coincides with the notion of inquisitiveness we used to have: a formula is inquisitive if and only if it has more than one maximal possibility. The reason why we introduced the notion in slightly different terms is that this presentation will also make perfect sense in the first-order case. The characterization in terms of maximal possibilities does not carry over to that setting, where —as we shall see— a formula can raise issues whose resolutions are not alternative but form an infinite chain, like in the case of the boundedness example. Nevertheless, the equivalent of proposition 6.2.20 will still hold.

Observe that item (4) shows that the support conditions of a formula are sufficient to determine whether or not it is inquisitive, although they do not determine its resolutions. In particular, just like informativeness, inquisitiveness is invariant under equivalence.

So far, no surprises: the class of formulas that are informative and inquisitive are solid enough. For formulas whose proposal only contains maximal possibilities, the new system behaves exactly like the system discussed in chapter 2, of which we can safely claim to have a good understanding. But how to interpret formulas that *do* propose non-maximal possibilities? What role do the additional possibilities play in their meaning?

Unfortunately, this is a question to which we do not have a certain answer. We suggest a natural interpretation for non-maximal possibilities in terms of *might*-suggestions put forward by a formula. Whether or not the actual behaviour of the semantics matches this hypothesis is something that will require further consideration; a little attempt to gather some indications on this question will be made in section 6.4, but the results will not prove to be conclusive in either direction.

Let us move on to explain our suggestion. Consider for instance the formula

$p \vee (p \wedge q)$, whose meaning is depicted in figure 6.1(a). It is clear that this formula informs that p is the case. It is not inquisitive, since it does not raise an issue: if $p \vee (p \wedge q)$ is true, it must be true because p is true. As far as information and issues are concerned, $p \vee (p \wedge q)$ is exactly equivalent to p.

Still, $p \vee (p \wedge q)$ differs from p in that it also proposes the possibility to update with $p \wedge q$ —without supplying any information about it— thus allowing other dialogue participants to compliantly react by asserting $p \wedge q$ if they can.

Our hypothesis is that the proposal made by $p \vee (p \wedge q)$ may be understood as follows: "I assure you that we can update with p; additionally, I would like to establish $p \wedge q$, but do not have enough information to do so". If so, the meaning of this formula could correspond to the effect of the natural language sentences: "p, and it might be that q as well", or "p, and perhaps q".

In view of this interpretation, we refer to non-maximal possibilities as *suggestions* and say that a formula is *suggestive* if it proposes one or more suggestions. In fact, we define suggestive formulas in slightly different terms.

Definition 6.2.21 (Suggestiveness)**.** We call a formula φ *suggestive* in case it has a possibility s that is strictly included in a *maximal* possibility t. Such a possibility s will be called a *suggestion*, or a *highlight* of φ.

Since in the propositional setting any possibility is included in a maximal one, this amounts to saying that a formula is suggestive if it has non-maximal possibilities. However, this definition has the advantage of being appropriate in the first-order case as well, where non-maximal possibilities need not be suggestions: they may also form an integral part of a real issue (think of the boundedness example), and thus they serve a different purpose than being suggestions. We will come back to this issue in section 6.3.4.

Observe that differently from informativeness and inquisitiveness, the notion of suggestiveness is intensional, that is, not definable in terms of support alone and thus not invariant under equivalence.

The following proposition indicates that the three aspects we identified exhaust the meaning of a formula.

Proposition 6.2.22. If a formula φ is neither informative, nor inquisitive, nor suggestive, then $\varphi \sim \top$.

Proof. If φ is not suggestive, then all of its possibilities are maximal. If φ is also not inquisitive, it has only one maximal possibility, which must coincide with its truth-set. Finally if φ is not informative, its truth-set must be $\mathcal{I}$, so $\varphi \sim \top$. □

6.2.6 Assertions, questions, and conjectures

We used to define assertions as non-inquisitive formulas, and question as non-informative ones. That is, speaking in informal but evocative terms, assertions were formulas whose meaning lay entirely on the axis described by empty inquisitive component, and questions formulas whose meaning lay on the axis of empty informative component (see figure 2.1.3). This will be similar now, but instead of a plane we will have a cube and one more type of formulas. Assertions will be formulas that are purely informative, i.e. non-inquisitive and non-suggestive; questions will be purely inquisitive, i.e. non-informative and non-suggestive; finally, *conjectures* will be purely suggestive, i.e. non-informative and non-inquisitive.

Let us start considering the notion of assertion. We want assertion to be formulas whose only effect is (at most) to provide information.

Definition 6.2.23 (Assertions)**.** An *assertion* is a formula that is neither inquisitive nor suggestive.

The following proposition insures, among other things, that the definition given here coincides with the one given above, of assertions as formulas with a one-piece proposition. Incidentally, notice that corollaries 6.2.6, 6.2.7, and 6.2.8 supply many examples of assertions.

Proposition 6.2.24 (Alternative characterizations of assertions)**.** For any formula φ, the following are equivalent:

1. φ is an assertion;
2. φ has only one possibility;
3. $[\![\varphi]\!] = \{|\varphi|\}$;
4. $\varphi \sim\, !\varphi$.

Proof.

(1) $\Rightarrow$ (2) If φ is an assertion, then it is not inquisitive, so it has only one maximal possibility, and it is also not suggestive, so all its possibilities are maximal; therefore φ has only one possibility.

(2) $\Rightarrow$ (3) Follows from the equality $\bigcup[\![\varphi]\!] = |\varphi|$ (corollary 6.2.4).

(3) $\Rightarrow$ (1) Immediate by definition of inquisitiveness and suggestiveness.

(3) $\Leftrightarrow$ (4) Follows from corollary 6.2.8.

□

Analogously, questions will be formulas whose sole effect is (at most) to raise an issue.

Definition 6.2.25 (Questions)**.** A *question* is a formula that is neither informative nor suggestive.

Note that now a formula of the shape $?\varphi$ will not in general be a question: it will be if and only if φ is non-suggestive. However, if χ is a meaningful assertion (that is, neither tautological nor contradictory) then $?\chi$ does express the polar question whether χ.

Finally, we turn to the notion of *conjecture*. A conjecture is a formula whose only purpose is (at most) to highlight one or more possible updates, but without either providing or requiring any information.

Definition 6.2.26 (Conjectures)**.** A *conjecture* is a formula that is neither informative nor inquisitive.

To gather some intuition, let us look at figure 6.1(b). The formula $p \vee \top$, which we abbreviate as $\Diamond p$, has the effect of suggesting, or highlighting, the possibility that p, but without providing any information in regards. As such, it can be taken as a formal counterpart of the natural language sentences "it might be that p" or "perhaps p".

In general, for any assertion χ, the formula $\Diamond\chi = \chi \vee \top$ does not provide information and does not raise issues, but simply "highlights" the possibility that χ so that it can be compliantly picked by the other participants if they wish; thus, we think of $\Diamond\chi$ as "it might be that χ".

In the previous section we said that the proposition of $p \vee (p \wedge q)$ can be taken to represent the natural language utterances "p, and it might be that q", or "p, and perhaps q". Now this comes out of our interpretation of $\Diamond$: for, it is easy to see that $p \vee (p \wedge q) \sim p \wedge \Diamond q$. In general, if both χ and ξ are assertions and ξ (classically) entails χ, then $\chi \vee \xi \sim \chi \wedge \Diamond\xi$, so at least in this case we have a natural interpretation of the role of non-maximal possibilities as *might*-possibilities.

More in general, for an arbitrary formula φ, $\Diamond\varphi$ has the effect of highlighting the resolutions of φ without providing the information that φ is true and without necessarily *requiring* a resolution of φ.

In order to understand better the last remark, let us consider the difference between the polar question $?p$ and the formula $\Diamond p \vee \Diamond\neg p$ whose meaning is depicted in figure 6.1(c). The polar question $?p$ introduces in the conversation an issue that is only resolved by asserting either p or $\neg p$. The conjecture $\Diamond p \vee \Diamond\neg p$, on the other hand, does not introduce an issue: it gives the other participants the possibility to assert p and $\neg p$, but does not *require* this information, since it is also resolved by not uttering anything (or nodding, or saying "Ok", or whatever non-informative, non-inquisitive and non-suggestive conversational move $\top$ is taken to represent).

Proposition 6.2.27 (Alternative characterizations of conjectures)**.** The following are equivalent:

φ is a conjecture;

$\mathcal{I} \in [\![\varphi]\!]$;

$\varphi \sim \Diamond\varphi$;

φ is supported everywhere.

Proof.

(1) $\Rightarrow$ (2) If φ is a conjecture, then it is not inquisitive, so it has a greatest possibility, and it is also not informative, so this greatest possibility must coincide with $\mathcal{I}$.

(2) ⇒ (3) Follows from the fact that $[\![\Diamond\varphi]\!] = [\![\varphi \vee \top]\!] = [\![\varphi]\!] \cup [\![\mathcal{I}]\!]$.

(3) ⇒ (4) If $\varphi \sim \Diamond\varphi$ then $\varphi \equiv \Diamond\varphi = \varphi \vee \top \equiv \top$, so φ is supported everywhere.

(4) ⇒ (1) Immediate.

□

Observe that, differently from assertions and questions, conjectures admit a characterization in terms of support.

Also, notice the following funny fact: item (4) states that a formula is a conjecture if and only if it used to be a tautology in the maximalization semantics. In chapter 2 we refined the classical notion of meaning and we got a new class of meanings, what we called then questions, out of formulas that were classical tautologies; now we refined the notion of meaning even further and we obtained a new class of meanings, namely conjectures, out of what used to be the tautologies.

Finally, we remark three closure properties of the class of conjectures.

Remark 6.2.28. For any formulas φ and ψ,

1. $\Diamond\varphi$ is a conjecture;
2. if φ and ψ are conjectures, then so is $\varphi \wedge \psi$;
3. if at least one of φ and ψ is a conjecture, so is $\varphi \vee \psi$;
4. if ψ is a conjecture, then so is $\varphi \to \psi$.

Proof. $\mathcal{I} \in [\![\top]\!] \subseteq [\![\Diamond\varphi]\!]$. If $\mathcal{I} \in [\![\varphi]\!]$ and $\top \in [\![\psi]\!]$, then $\mathcal{I} = \mathcal{I} \cap \mathcal{I}$ is in $[\![\varphi \wedge \psi]\!]$ by definition of propositions. If $\mathcal{I} \in [\![\varphi]\!]$ or $\mathcal{I} \in [\![\psi]\!]$, then $\mathcal{I} \in [\![\varphi]\!] \cup [\![\psi]\!] = [\![\varphi \vee \psi]\!]$. Finally if $\mathcal{I} \in [\![\psi]\!]$ then letting $f^{\top} : [\![\varphi]\!] \to [\![\psi]\!]$ be the function mapping each $s \in [\![\varphi]\!]$ to $\mathcal{I}$ we have $\mathcal{I} = \Pi_{f^{\top}} \in [\![\varphi \to \psi]\!]$. □

This is in tune with the intuition that utterances like "it might be that p and it might be that q", "p, or it might be that q" and "if p, it might be that q" are all conjectures.

Note on the empty set. In possibility semantics, it is often the case that the empty, inconsistent state shows up among the possibilities for a formula. Obviously, in a conversation, it is never a meaningful option to choose to update the common ground to the inconsistent state. Therefore, in my opinion, the empty state should simply be disregarded when judging the proposal made by a formula.

Having the empty state in the semantics is handy for several purposes, in the first place to make the definition of propositions come out right, in particular as regards negations. But for practical purposes, as soon as a formula has at least another possibility, we should probably *not* regard the inconsistent state as a suggestion of the formula. We should thus also consider two formulas strongly equivalent if one only differs from the other only for the presence of the empty state.

There are plenty of examples where this appears in fact to be the reasonable thing to do. For instance, we would expect the formula $\Diamond p \wedge \Diamond\neg p$ to highlight the possibility that p and the possibility that $\neg p$, thus being strongly equivalent to $\Diamond p \vee \Diamond\neg p$: this is indeed what we obtain if we disregard the presence of the empty set in $[\![\Diamond p \wedge \Diamond\neg p]\!]$. We will see more convincing examples of this need in the first-order case.

6.2.7 Axiomatizing strong entailment

In this section we will give a syntactic characterization of the notion of strong entailment. Let us start from a survey of logical principles that do and do not hold under this interpretation.

Remark 6.2.29 (Valid and invalid logical laws under strong entailment)**.**

1. The connectives $\wedge$ and $\vee$ are commutative and associative: $\varphi\wedge\psi \sim \psi\wedge\varphi$, $(\varphi \wedge \psi) \wedge \chi \sim \varphi \wedge (\psi \wedge \chi)$, and the same for $\vee$.

2. Disjunction is idempotent. However, conjunction is *not* idempotent: $\varphi \mid\!\!\mid\!\!\sim \varphi \wedge \varphi$ but in general not viceversa; for a counterexample, take $\varphi = p \vee q$: $p \wedge q$ is a resolution for $(p \vee q) \wedge (p \vee q)$ but not for $p \vee q$.

3. Conjunction distributes over disjunction: $\varphi \wedge (\psi \vee \chi) \sim (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$. However, the other distribution law holds only in one direction: $\varphi \vee (\psi \wedge \chi) \mid\!\!\mid\!\!\sim (\varphi \vee \psi) \wedge (\varphi \vee \chi)$ but in general not viceversa; for a counterexample, take $\varphi = p$ and $\psi = \chi = q$: $p \wedge q$ is a resolution for $(p \vee q) \wedge (p \vee q)$ but not for $p \vee (q \wedge q)$.

4. Bottom and top interact with conjunction as usual: $\varphi \wedge \bot \sim \bot$ and $\varphi \wedge \top \sim \varphi$. However, the same does not hold for disjunction: $\varphi \mid\!\!\mid\!\!\sim \varphi \vee \bot$ and $\varphi \mid\!\!\mid\!\!\sim \varphi \vee \top$ but, in general not viceversa; for a counterexample, take $\varphi = p$: then $\bot$ is a resolution for $p \vee \bot$ and $\top$ is a resolution for $p \vee \top$, but neither is a resolution for p.

5. Neither direction of the deduction theorem holds. For instance $(p \to \top) \sim \top$ but it is not the case that $p \mid\!\!\mid\!\!\sim \top$, and $p \mid\!\!\mid\!\!\sim p \vee \neg p$ but $(p \to p \vee \neg p) \not\sim \top$. It *is* however the case that if $\varphi \mid\!\!\mid\!\!\sim \psi$ then $\top \mid\!\!\mid\!\!\sim \varphi \to \psi$.

6. If φ and ψ have no common possibility, then $(\varphi \vee \psi) \to \chi \sim (\varphi \to \chi) \wedge (\psi \to \chi)$.

7. If φ is an assertion, then $\varphi \to (\psi \vee \chi) \sim (\varphi \to \psi) \vee (\varphi \to \chi)$.

8. If φ and ψ are assertions, then $\varphi \sim \psi$ iff φ and ψ are classically equivalent.

We will characterize strong entailment syntactically by a derivation system having *no axioms*, but only a set of derivation rules. We will write $\varphi \mid\!\sim \psi$ in case there exists a derivation of ψ from φ according to the rules, and $\varphi \sim\!\mid\!\sim \psi$ in case $\varphi \mid\!\sim \psi$ and $\psi \mid\!\sim \varphi$.

Definition 6.2.30 (Rules)**.** We list below the rules of our derivation system. Rules formulated with a double line can be used in both directions, whereas rules with a single line can be used only downwards.

1. **Commutativity:** $\dfrac{\varphi\wedge\psi}{\psi\wedge\varphi}$ and $\dfrac{\varphi\vee\psi}{\psi\vee\varphi}$
2. **Associativity:** $\dfrac{(\varphi\wedge\psi)\wedge\chi}{\varphi\wedge(\psi\wedge\chi)}$ and $\dfrac{(\varphi\vee\psi)\vee\chi}{\varphi\vee(\psi\vee\chi)}$
3. **Disjunction-free substitution:** if χ is a disjunction-free subformula of φ and φ^* is obtained from φ by replacing χ (not necessarily every occurrence of χ) with a classically equivalent disjunction-free formula χ', then $\dfrac{\varphi}{\varphi^*}$
4. **Distribution of $\wedge$ over $\vee$:** $\dfrac{\varphi\wedge(\psi\vee\chi)}{(\varphi\wedge\psi)\vee(\varphi\wedge\chi)}$
5. **Distribution of disjunction in the antecedent of an implication:** if φ and ψ are disjunction-free and not classically equivalent, $\dfrac{(\varphi\vee\psi)\to\chi}{(\varphi\to\chi)\wedge(\psi\to\chi)}$
6. **Distribution of disjunction in the consequent of an implication:** if φ is disjunction-free, $\dfrac{\varphi\to(\psi\vee\chi)}{(\varphi\to\psi)\vee(\varphi\to\chi)}$
7. **Substitution of interderivables:** if we have already shown that $\chi \dashv\vdash \chi'$ and $\varphi[{\chi'}/{\chi}]$ denotes the formula obtained from φ by replacing any occurrence of χ by χ', then $\dfrac{\varphi}{\varphi[{\chi'}/{\chi}]}$
8. **Addition of disjuncts:** $\dfrac{\varphi}{\varphi\vee\psi}$

In the following discussion, rules (1) and (2) will always be used tacitly, by disregarding the order and the parsing of conjunctions and disjunctions. Observe that since all the rules but the last are reversible, if φ was derived from ψ without using rule (8), then reversing the derivation yields a derivation of ψ from φ.

The following theorem states that the given derivation system soundly and completely axiomatizes strong entailment.

Theorem 6.2.31 (Soundness and completeness of $\vdash$ for strong entailment)**.** For any formulas φ and ψ, $\varphi \models \psi \iff \varphi \vdash \psi$.

We start by proving the soundness of the system.

Lemma 6.2.32 (Soundness)**.** If $\varphi \vdash \psi$ then $\varphi \models \psi$.

Proof. Consider a formula φ: we show by induction that any formula occurring in a $\vdash$ −derivation from φ is strongly entailed by φ.

Since our derivation system has no axioms, the first formula in the derivation must be φ, and obviously $\varphi \models \varphi$. Now, according to remark 6.2.29, any rule but the last one turns a formula into a strongly equivalent one; so, if $\varphi \models \chi$ and

χ' is obtained from χ by an application of one of the rules (1)-(7), then $\varphi \mid\!\sim \chi'$ by the transitivity of $\mid\!\sim$. Finally, if $\varphi \mid\!\sim \chi$ and $\chi \vee \xi$ is obtained from χ by an application of rule (8), then $[\![\varphi]\!] \subseteq [\![\chi]\!] \subseteq [\![\chi]\!] \cup [\![\xi]\!] = [\![\chi \vee \xi]\!]$. □

In chapter 3 we saw that an easy way to prove completeness for an axiomatization of inquisitive entailment passes through the analysis of the ingredients needed to justify the disjunctive-negative translation of a formula.

Analogously, we will get to completeness for strong entailment through the following lemma which shows that the rules given above are sufficient to justify the strong equivalence between a formula φ and the disjunction of assertions $\bigvee \mathcal{R}(\varphi)$.

Lemma 6.2.33. For any φ, $\varphi \sim\!\mid\!\sim \bigvee \mathcal{R}(\varphi)$.

Proof. By induction on φ. The basic cases are trivial.

1. Consider a disjunction $\varphi \vee \psi$. The induction hypotheses guarantees that $\varphi \sim\!\mid\!\sim \bigvee \mathcal{R}(\varphi)$ and $\psi \sim\!\mid\!\sim \bigvee \mathcal{R}(\psi)$; so, starting from $\varphi \vee \psi$, two applications of rule (7) yield $\bigvee \mathcal{R}(\varphi) \vee \bigvee \mathcal{R}(\varphi)$; but $\bigvee \mathcal{R}(\varphi) \vee \bigvee \mathcal{R}(\psi) = \bigvee \mathcal{R}(\varphi \vee \psi)$.

   This shows that $\varphi \vee \psi \mid\!\sim \bigvee \mathcal{R}(\varphi \vee \psi)$. The converse is obtained by reversing the derivation, since rule (8) was not used.

2. Consider a conjunction $\varphi \wedge \psi$. Again using the induction hypotheses on both φ and ψ and applying rule (7), from $\varphi \wedge \psi$ we get $(\bigvee \mathcal{R}(\varphi)) \wedge (\bigvee \mathcal{R}(\psi))$; then, using the distribution rule (4) we obtain $\bigvee_{\rho \in \mathcal{R}(\varphi), \sigma \in \mathcal{R}(\psi)} (\rho \wedge \sigma)$. Now, for any $\rho \in \mathcal{R}(\varphi)$ and any $\sigma \in \mathcal{R}(\psi)$, the formula $\rho \wedge \sigma$ and its normal form $(\rho \wedge \sigma)_{nf}$ are classically equivalent disjunction-free formulas. Therefore, applying rule (3) for each disjunct we obtain $\bigvee_{\rho \in \mathcal{R}(\varphi), \sigma \in \mathcal{R}(\psi)} (\rho \wedge \sigma)_{nf}$, which amounts to $\bigvee \mathcal{R}(\varphi \wedge \psi)$.

   Hence, $\varphi \wedge \psi \mid\!\sim \bigvee \mathcal{R}(\varphi \wedge \psi)$. The converse is obtained again by reversing the derivation.

3. Consider an implication $\varphi \to \psi$. By the induction hypotheses and rule (7), from $\varphi \to \psi$ we obtain $\bigvee \mathcal{R}(\varphi) \to \bigvee \mathcal{R}(\psi)$. Now, by definition all the elements of $\mathcal{R}(\varphi)$ are non-classically equivalent disjunction free formulas; therefore, using rule (5) we can distribute the disjunction in the antecedent over the implication, obtaining $\bigwedge_{\rho \in \mathcal{R}(\varphi)} (\rho \to \bigvee \mathcal{R}(\psi))$.

   Then consider any $\rho \in \mathcal{R}(\varphi)$: since ρ is disjunction-free, starting from $\rho \to \bigvee \mathcal{R}(\psi)$ we can use rule (6) to distribute the disjunction in the consequent over the implication, obtaining $\bigvee_{\sigma \in \mathcal{R}(\psi)} (\rho \to \sigma)$. Conversely, the other direction of rule (6) allows us to derive $\rho \to \bigvee \mathcal{R}(\psi)$ from $\bigvee_{\sigma \in \mathcal{R}(\psi)} (\rho \to \sigma)$. Thus, for any $\rho \in \mathcal{R}(\varphi)$ we have $\rho \to \bigvee \mathcal{R}(\psi) \sim\!\mid\!\sim \bigvee_{\sigma \in \mathcal{R}(\psi)} (\rho \to \sigma)$.

   Hence, we can apply rule (7) for each ρ to $\bigwedge_{\rho \in \mathcal{R}(\varphi)} (\rho \to \bigvee \mathcal{R}(\psi))$ and obtain $\bigwedge_{\rho \in \mathcal{R}(\varphi)} \bigvee_{\sigma \in \mathcal{R}(\psi)} (\rho \to \sigma)$. Applying again the distribution rule (4) for conjunction, we obtain $\bigvee_{f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)} \bigwedge_{\rho \in \mathcal{R}(\varphi)} (\rho \to f(\rho))$. Now, since all resolutions are disjunction-free, each formula $\rho \to f(\rho)$ is disjunction-free and classically equivalent to its normal form $(\rho \to f(\rho))_{nf}$, which is also disjunction-free. Therefore, a number of applications of rule (3) yields the formula $\bigvee_{f:\mathcal{R}(\varphi)\to\mathcal{R}(\psi)} \bigwedge_{\rho\in\mathcal{R}(\varphi)}(\rho \to f(\rho))_{nf}$, which is simply $\bigvee \mathcal{R}(\varphi \to \psi)$.

This shows $\varphi \to \psi \mid\!\sim \bigvee \mathcal{R}(\varphi \to \psi)$. As usual, the converse is proved by the reverse derivation.

□

Proof of theorem 6.2.31. Suppose $\varphi \mid\mid\!\sim \psi$. By the previous lemma, from φ we can derive $\bigvee \mathcal{R}(\varphi)$. Now, since $[\![\varphi]\!] \subseteq [\![\psi]\!]$, for any $\rho \in \mathcal{R}(\varphi)$ there is a classically equivalent formula $\sigma_\rho \in \mathcal{R}(\psi)$. Since resolutions are always disjunction-free, we can use the disjunction-free substitution rule to turn $\bigvee \mathcal{R}(\varphi)$ into $\bigvee_{\rho\in\mathcal{R}(\varphi)}\sigma_\rho$.

But note that $\bigvee \mathcal{R}(\psi)$ is nothing but $\bigvee_{\rho\in\mathcal{R}(\varphi)}\sigma_\rho$ with some extra disjuncts, and thus it can be derived from the latter through several applications of the rule of addition of disjuncts. Finally we can use the previous lemma once again to derive ψ from $\bigvee \mathcal{R}(\psi)$. □

6.3 First-order possibility semantics

6.3.1 Propositions

Equipped with our new approach to the meaning of formulas which does not rely on maximality issues anymore, we can now return to the task of defining a satisfactory inquisitive semantics for a first-order language $\mathcal{L}$. Recall from section 6.1 that we assume a fixed structure $\mathbb{D}$ consisting of a domain and an interpretation of the function symbols in $\mathcal{L}$: our semantics is then concerned with models for $\mathcal{L}$ based on $\mathbb{D}$, and $\mathcal{I}$ denotes the set of all such models.

Obviously, in order to define truth, support, possibilities etc. for a formula we need to know what elements of D the free variables in the formula refer to; for this reason, all the notions introduced in this section are relativized to an assignment g.

As usual, if $\varphi = \varphi(x_1, \ldots, x_n)$ is a formula whose free variables are among $x_1, \ldots, x_n$ and if $d_1, \ldots, d_n$ are elements of D, we will also write things such as $[\![\varphi(d_1, \ldots, d_n)]\!]$, or "$\varphi(d_1, \ldots, d_n)$ is an assertion" to mean, respectively, $[\![\varphi]\!]_g$ and "φ is an assertion relative to g" where g is any assignment mapping each x_i to d_i.

Having remarked this, we are ready to introduce the semantics. The obvious analogous of definition 6.2.1 is the following.

Definition 6.3.1 (Propositions)**.** Define the proposition associated to a formula relative to an assignment g as follows:

1. $[\![\varphi]\!]_g = \{|\varphi|_g\}$ if φ is atomic;
2. $[\![\bot]\!]_g = \{\emptyset\}$;
3. $[\![\varphi \vee \psi]\!]_g = [\![\varphi]\!]_g \cup [\![\psi]\!]_g$;

4. $[\![\varphi \wedge \psi]\!]_g = \{s \cap t \mid s \in [\![\varphi]\!]_g \text{ and } t \in [\![\psi]\!]_g\}$;
5. $[\![\varphi \to \psi]\!]_g = \{\Pi_f \mid f : [\![\varphi]\!]_g \to [\![\psi]\!]_g\}$, where

$$\Pi_f = \{w \in I \mid \text{ for all } s \in [\![\varphi]\!]_g, \text{ if } w \in s \text{ then } w \in f(s)\}$$

6. $[\![\exists x \varphi]\!]_g = \bigcup_{d \in D} [\![\varphi]\!]_{g[x \mapsto d]}$;
7. $[\![\forall x \varphi]\!]_g = \{\bigcap_{d \in D} s_d \mid s_d \in [\![\varphi]\!]_{g[x \mapsto d]} \text{ for all } d \in D\}$;

It is easy to check by induction that, with this definition of possibilities, we have the desired connection between support and possibilities.

Proposition 6.3.2. For any assignment g, any state s and any formula φ we have

$$s, g \models \varphi \iff s \subseteq t \text{ for some } t \in [\![\varphi]\!]_g$$

As usual, this comes with the associated corollary that information is treated classically.

Corollary 6.3.3 (Classical treatment of information)**.** For any assignment g and any formula φ,

$$\bigcup [\![\varphi]\!]_g = |\varphi|_g$$

Proof. If $s \in [\![\varphi]\!]_g$ then by the previous proposition $s, g \models \varphi$ and therefore, by persistence and the classical behaviour of singletons, $s \subseteq |\varphi|_g$. Hence, $\bigcup [\![\varphi]\!]_g \subseteq |\varphi|_g$. Conversely, if $M \in |\varphi|_g$, then by the classical behaviour of singletons and the previous proposition M must be contained in some $s \in [\![\varphi]\!]_g$ and therefore it must belong to $\bigcup [\![\varphi]\!]_g$. □

Of course –unlike in the propositional setting– now the proposition associated to a formula may well consist of infinitely many possibilities in case the domain D is infinite, as the recursive clause for the existential quantifier shows. However, it is immediate to verify inductively that, with the caveat that infinite cardinalities may be involved, the analogue of Groenendijk's inequalities still holds.

Proposition 6.3.4. For any assignment g,

1. $\#[\![p]\!]_g = \#[\![\bot]\!]_g = 1$
2. $\#[\![\varphi \vee \psi]\!]_g \leq \#[\![\varphi]\!]_g + \#[\![\psi]\!]_g$
3. $\#[\![\varphi \wedge \psi]\!]_g \leq \#[\![\varphi]\!]_g \#[\![\psi]\!]_g$
4. $\#[\![\varphi \to \psi]\!]_g \leq \#[\![\psi]\!]_g{}^{\#[\![\psi]\!]_g}$
5. $\#[\![\exists x \varphi]\!]_g \leq \Sigma_{d \in D} \#[\![\varphi]\!]_{g[x \mapsto d]}$
6. $\#[\![\forall x \varphi]\!]_g \leq \Pi_{d \in D} \#[\![\varphi]\!]_{g[x \mapsto d]}$

Say that a formula φ is an *assertion* relative to an assignment g in case its proposition $[\![\varphi]\!]_g$ consists of one sole possibility: again, assertions will be redefined later on in a different but equivalent way.

The previous inequalities yield the natural analogues of corollaries 6.2.6, 6.2.7, and 6.2.8, giving sufficient syntactic conditions for a formula to be an assertion.

Corollary 6.3.5. For any assignment g, any propositional letter p and any formulas φ and ψ, the following holds:

1. p is an assertion relative to g;
2. $\bot$ is an assertion relative to g;
3. if both φ and ψ are assertions relative to g, then so is $\varphi \wedge \psi$;
4. if ψ is an assertion relative to g, then so is $\varphi \to \psi$;
5. if φ is an assertion relative to $g[x \mapsto d]$ for all $d \in D$, then $\forall x \varphi$ is an assertion relative to g.

Corollary 6.3.6. A formula that contains neither disjunction nor the existential quantifier is an assertion relative to any assignment.

Thus, disjunction and the existential quantifier are the only sources of non-classical behaviour in the first-order semantics.

Corollary 6.3.7. A negation is an assertion relative to any assignment.

6.3.2 Entailment and strong entailment

Just like in the propositional case, it is possible to identify two different notions of entailment.

Definition 6.3.8 (Entailment and strong entailment)**.** Let φ and ψ be two formulas. Relative to a fixed assignment g, we say that:

1. φ *entails* ψ, in symbols $\varphi \models_g \psi$, in case for any state s, if $s, g \models \varphi$ then $s, g \models \varphi$;
2. φ *strongly entails* ψ, in symbols $\varphi \mid\!\!\sim_g \psi$, in case $[\![\varphi]\!]_g \subseteq [\![\psi]\!]_g$;
3. φ and ψ are *equivalent*, in symbols $\varphi \equiv_g \psi$, in case $\varphi \models_g \psi$ and $\psi \models_g \varphi$;
4. φ and ψ are *strongly equivalent*, in symbols $\varphi \sim_g \psi$, in case $\varphi \mid\!\!\sim_g \psi$ and $\psi \mid\!\!\sim_g \varphi$.

It is obvious by proposition 6.3.2 that strong entailment implies entailment, and strong equivalence implies equivalence.

Remark 6.3.9. For any assignment g, if $\varphi \mid\!\!\sim_g \psi$ then $\varphi \models_g \psi$.

The difference between the two relations is the same as we discussed for the propositional case: $\varphi \models_g \psi$ expresses the fact that ψ is resolved whenever φ is, that is, that a resolution for ψ can always be obtained from a resolution of φ. The relation $\varphi \mid\mid\!\sim_g \psi$, on the other hand, means that resolutions for φ themselves are also resolutions for ψ: whenever φ is resolved, ψ is resolved *in the same way*.

This difference is well illustrated by the following example.

Example 6.3.10. We will consider two variations of the boundedness formula that do not share any possibility and still are equivalent, since a resolution of either of them can always be converted into a resolution of the other.

Consider the language consisting of a binary function symbol $+$, a constant 1 and a predicate symbol P. Our structure $\mathbb{D}$ consists of the natural numbers with the standard interpretation of $+$ and 1. We will make use of the following abbreviations:

1. $x \leq y$ for $\exists z(x + z = y)$
2. $E(x)$ for $!\exists z(z + z = x)$
3. $O(x)$ for $!\exists z(z + z + 1 = x)$

It is clear that $E(x)$ and $O(x)$ are assertions meaning, respectively, that x is even, and that x is odd. Then, like in section 6.1, let $B(x) = !\forall y(Py \to y \leq x)$: by corollary 6.3.7, $B(x)$ is an assertion stating that x is an upper bound for P. Now define sentences B_E and B_O as follows:

1. $B_E = \exists x(E(x) \wedge B(x))$;
2. $B_O = \exists x(O(x) \wedge B(x))$;

In words, the sentences in question represent: "There is an even upper bound to P" and "There is an odd upper bound to P". If we compute the propositions associated with these sentences, we obtain:

1. $[\![B_E]\!] = \{|B(2n)| \mid n \in \omega\}$;
2. $[\![B_O]\!] = \{|B(2n + 1)| \mid n \in \omega\}$.

Thus, B_E and B_O invite different resolutions: indeed, they have no common resolution at all, since $|B(n)| \neq |B(m)|$ for $n \neq m$. This is in tune with the fact that it would not be compliant to respond, say, "Oh, yes, seven!" to the utterance "There is an even upper bound to P".

However, $B_E \equiv B_O$. For, if $s \models B_E$, then it must be $s \subseteq |B(2n)| \subseteq |B(2n+1)|$ for some number n by proposition 6.2.2, whence $s \models B_O$. Viceversa, if for some n we have $s \subseteq |B(2n + 1)|$, then $s \subseteq |B(2n + 2)|$ and so $s \models B_E$.

The point is that given an even upper bound to P we can obviously derive an odd upper bound by adding one, and viceversa; thus, a resolution for one of B_E, B_O can always be deduced from a resolution of the other.

From the point of view of support, which describes what information is needed to resolve a formula, the two formulas are the same. Incidentally, note that also from the point of view of the responder(s), B_E and B_O are equally hard to solve, so there is really no point in uttering B_O after B_E has already been uttered (or viceversa).

On the other hand, what we try to model via possibilities is the effect of utterances in a dialogue. In this respect, B_E and B_O are certainly different, since they invite different responses. What is compliant with B_O need not be compliant with B_E, as the example of "Oh, yes, seven!" shows.

The very same difference had been encountered before in Groenendijk's partition semantics for *who* questions. Assuming that the relation between the time in London and the time in Amsterdam is common knowledge, the two questions "What time is it in London?" and "What time is it in Amsterdam?" produce the same partition of the space of models, and the information needed in order to answer them is exactly the same. Nonetheless, the two questions invite different answers and therefore their utterance produces a different effect in a conversation.

This discussion should hopefully clarify in what sense we may talk of support, entailment, and equivalence as the extensional counterpart of propositions, strong entailment, and strong equivalence.

6.3.3 Resolutions

In the previous section we saw that in the propositional setting, any possibility s for a formula φ could be expressed by a formula ρ_s that we called a *resolution* of φ.

This is clearly not the case in the first-order setting (unless the domain is finite and there are names for each element). For instance, suppose we are in the setting of the previous example and consider the formula $(?x)P(x) = \forall x?P(x)$.

It is easy to check that according to our semantics, this formula represents the question "which numbers have the property P?", being associated with the proposition $[\![\forall x?P(x)]\!] = \{\{M_Y\} \mid Y \subseteq \omega\}$ where M_Y is the $\mathbb{D}$-model defined by $P^M = Y$.

Since the natural numbers have uncountably many subsets, the formula in question has uncountably many possibilities, and since the language we are considering is countable, there will be formulas that cannot be expressed.

However, in order to better denote and understand possibilities, it might be convenient to simply *create* all resolutions for a formula by adding constants $\overline{d}$ for each element $d \in D$ and by allowing infinite conjunctions and disjunctions (actually, it is only the former that we need, but the latter will come in handy as well). Truth-sets for this extended language are defined in the obvious way, and as for propositions, one just needs to allow the union mentioned in the disjunction clause and the intersection mentioned in the conjunction clause to be infinite. Note that in this language we do not need the quantifiers anymore, since we can identify $\forall x\varphi(x)$ and $\exists x\varphi(x)$ with the strongly equivalent formulas $\bigwedge_{d\in D}\varphi(\overline{d})$ and $\bigvee_{d\in D}\varphi(\overline{d})$, and so we can also get rid of variables.

Again, we need to insure against multiple copies. We do this by fixing a normal form $(\ )_{nf}$ for classical logic in which formulas are represented in a disjunction-free and existential-free form; we also assume that this normal form leaves atoms and $\bot$ unchanged. We can then define resolutions as follows.

Definition 6.3.11 (Resolutions)**.**

1. $\mathcal{R}(\varphi) = \{\varphi\}$ if φ is atomic;
2. $\mathcal{R}(\bot) = \{\bot\}$;
3. $\mathcal{R}(\bigvee_{j\in J}\varphi_j) = \bigcup_{j\in J}\mathcal{R}(\varphi_j)$;
4. $\mathcal{R}(\bigwedge_{j\in J}\varphi_j) = \{(\bigwedge_{j\in J}\rho_j)_{nf} \mid \rho_j \in \mathcal{R}(\varphi_j)\}$;
5. $\mathcal{R}(\varphi \to \psi) = \{(\bigwedge_{\rho\in\mathcal{R}(\varphi)}(\rho \to f(\rho)))_{nf} \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$.

Since we required resolutions to be disjunction-free and existential-free, resolutions are assertions by (an infinitary analogue of) corollary 6.3.6. The following proposition is nothing but an infinitary version of proposition 6.2.10, but we can apply it to formulas in the original language to obtain a syntactic representation of their proposition in terms of resolutions, which are now formulas in the extended language.

Proposition 6.3.12. For any formula φ, the map $\rho \to |\rho|$ from $\mathcal{R}(\varphi)$ to $[\![\varphi]\!]$ is a bijection. In particular, $[\![\varphi]\!] = \{|\rho| \mid \rho \in \mathcal{R}(\varphi)\}$.

Corollary 6.3.13. For any formula φ, $\varphi \sim \bigvee \mathcal{R}(\varphi)$.

Obviously, our focus will remain on the original finitary language, as we are interested in human communication; nevertheless, resolutions provide a very intuitive representation of the proposal associated to a formula. For instance, resolutions for the who-question $\forall x?P(x)$ considered above are —up to equivalence— precisely the full answers to the who-question, namely formulas of the shape $\bigwedge_{n\in\omega}W(\overline{n})$ where, for any $n \in \omega$, $W(\overline{n})$ is either $P(\overline{n})$ or $\neg P(\overline{n})$. Of course, most of these answers cannot be expressed in the finite language, but some can (provided that we are also equipped with the multiplication symbol, these will be precisely the arithmetic sets).

Of course, there are also many formulas whose resolutions can *all* be expressed in the finite language. For instance, resolutions for the boundedness formula are formulas of the shape $B(\overline{n})$ for $n \in \omega$, so they can be expressed in a language having terms for all natural numbers.

Additionally, if the domain D happens to be finite —which is the case in most dialogue situations— then our construction makes perfect sense *without* moving to an infinite language: we just need names for all individuals and the above definition will provide a notion of resolutions that works just as well as the propositional one.

6.3.4 Inquisitiveness, informativeness, suggestiveness

In this section we will follow the path we traced in the propositional case and see that the meaning of formulas can be analyzed along exactly the same lines, identifying an *informative*, an *inquisitive* and a *suggestive* potential.

Like in the previous section, relative to an assignment g we call a formula φ a *tautology* in case $[\![\varphi]\!]_g = \{\mathcal{I}\}$, and a *contradiction* in case $[\![\psi]\!]_g = \{\emptyset\}$. Thus, tautologies are formulas that make a trivial proposal, while contradictions make unacceptable proposals. It follows from the equality $\bigcup[\![\varphi]\!]_g = |\varphi|_g$ that a formula is a classical contradiction if and only if it is a contradiction on any structure $\mathbb{D}$ relative to any assignment g. Classical tautologies, on the other hand, need not be tautologies in the inquisitive sense.

The notion of informativeness is as usual unproblematic. Relative to an assignment g, a formula is *informative* if it proposes to eliminate some models; according to corollary 6.3.3, this amounts to the formula not being true on all $\mathbb{D}-$models under g.

Definition 6.3.14 (Informativeness)**.** Relative to an assignment g, a formula φ is *informative* in case $|\varphi|_g \neq \mathcal{I}$.

As usual, relative to any assignment, a formula is informative if and only if its negation is not a contradiction: for, a negation $\neg\varphi$ is simply an assertion which has the effect of denying the informative content of φ.

We say that a formula φ is *inquisitive* in case it *requires* information, that is, if it is only resolved by formulas that supply additional information. As we argued in the propositional case, this condition can be formulated as follows.

Definition 6.3.15 (Inquisitiveness)**.** Relative to an assignment g, a formula φ is *informative* in case $|\varphi|_g \notin [\![\varphi]\!]_g$.

The analogue of proposition 6.2.20 still holds and is proven in the same way.

Proposition 6.3.16 (Alternative characterization of inquisitiveness)**.** For any assignment g and any formula φ, the following are equivalent.

1. φ is not inquisitive relative to g;
2. φ is classically equivalent to one of its resolutions relative to g;
3. $[\![\varphi]\!]_g$ has a greatest element;
4. $|\varphi|, g \models \varphi$.
5. $\varphi \equiv\ !\varphi$

The only difference is that now an inquisitive formula need not necessarily propose *alternatives*, in the sense of several maximal possibilities; it may also propose an infinite chain of possibilities included in one another: think of the boundedness example, whose possibilities are $|B(0)| \subseteq |B(1)| \subseteq |B(2)| \subseteq \ldots$; as expected, it is inquisitive, because it can only be resolved by providing an upper bound for P.

The boundedness formula shows very clearly that in the first-order case, non-maximal possibilities may be needed in order to specify issues. At the same time, possibilities that are neither maximal –that is, alternatives– nor part of such an infinite-chain issue are always included in a maximal possibility, and thus they play the same role as suggestions in the propositional case.

Definition 6.3.17 (Suggestiveness)**.** Relative to an assignment g, a formula φ is *suggestive* in case it has a possibility s that is strictly included in a maximal possibility t. As usual, if this is the case we refer to such a possibility s as a *suggestion*, or a *highlight* of φ.

If a formula is not inquisitive, then it has a greatest possibility coinciding with its truth-set; if it is also not suggestive, then this must be its unique possibility; finally, if in addition the formula is not informative, then its truth-set must coincide with $\mathcal{I}$. This shows the following result indicating that there are no further dimensions to the meaning of a formula.

Remark 6.3.18. If, relative to an assignment g, a formula φ is neither informative, nor inquisitive, nor suggestive, then $\varphi \sim_g \top$.

Just like in the propositional case, inquisitiveness and informativeness are purely extensional notions, that is, they can be defined in terms of support alone. This may at first appear surprising in the case of inquisitiveness, since we saw that support is not sufficient to determine the issue raised by a formula, but it is not. For, by definition a formula is inquisitive in case it requires information from the other participants. Now, two formulas like B_E and B_O in example 6.3.10 may raise different issues, i.e. propose different possibilities, but as long as they are equivalent, the information needed to settle them is the same; thus, if one *requires* information from the other participants in order to be settled, so does the other.

Suggestiveness, on the other hand, is obviously not invariant under equivalence and therefore *not* definable in terms of support.

6.3.5 Assertions, questions, and conjectures

Like in the propositional case, we isolate three classes of formulas that serve only one of the three purposes identified in the previous section.

Definition 6.3.19 (Assertions)**.** Relative to an assignment g, a formula is an *assertion* if it is neither inquisitive nor suggestive.

Assertions admit the usual alternative characterizations; the proof is the same we gave for the propositional analogue, proposition 6.2.24.

Proposition 6.3.20 (Alternative characterizations of assertions)**.** For any assignment g and any formula φ, the following are equivalent:

1. φ is an assertion relative to g;

2. φ has only one possibility relative to g;
3. $[\![\varphi]\!]_g = \{|\varphi|_g\}$;
4. $\varphi \sim_g !\varphi$.

Corollaries 6.3.5, 6.3.6 and 6.3.7 provide many examples of assertions.

Definition 6.3.21 (Questions)**.** Relative to an assignment g, a formula is a *question* if it is neither informative nor suggestive.

Example 6.3.22. Consider the formula $(?x)\chi(x) = \forall x ?\chi(x)$, where $\chi(x)$ is an assertion relative to any assignment. Resolutions for this formula are complete answers to the question "for which x is χ true?". Accordingly, possibilities are of the shape $A_Y = \{M \in \mathcal{I} \mid \chi^M = Y\}$ for $Y \subseteq D$, where χ^M denotes the extension of χ in the model M.

Now, obviously if $\chi^M = Y$ then χ^M cannot be equal to any $Y' \neq Y$, so all the possibilities A_Y are disjoint. Moreover, any model M belongs to some possibility, namely to A_{χ^M}. Thus, the meaning of $(?x)\chi(x)$ forms a partition of the common ground into pieces corresponding to the complete answers to the who-question.

This means that $(?x)\chi(x)$ has precisely the same meaning that it used to have in the partition semantics of Groenendijk's logic of interrogation (Groenendijk, 1999; ten Cate and Shan, 2007). The same is true of the formula $(?x_1 \dots x_n)\chi = \forall x_1 \dots \forall x_n ?\chi$. The remarkable thing is that whereas in that system a purpose-made operator $(?x_1 \dots x_n)$ had to be explicitly defined, here we get it for free from the inquisitive behaviour of the standard logical connectives.

Who-questions provide another reason to choose, as discussed in the previous section, to disregard the empty state as a possibility and not to consider it a suggestion. For, who questions of the kind just discussed will produce a partition which might in general include the empty possibility. Still, we do not want this to prevent us from calling them questions.

Definition 6.3.23 (Conjectures)**.** Relative to an assignment g, a formula is a *conjecture* if it is neither informative nor inquisitive.

Arguing like in the previous section we obtain the following alternative characterization of conjectures.

Proposition 6.3.24 (Alternative characterization of conjectures)**.** For any assignment g and any formula φ, the following are equivalent:

1. φ is a conjecture relative to g;
2. $\mathcal{I} \in [\![\varphi]\!]_g$;
3. $\varphi \sim_g \Diamond\varphi$;
4. $\varphi \equiv_g \top$.

For instance, the formula $\exists x \Diamond P(x) \sim \Diamond \exists x P(x)$ is a conjecture that highlights, for each $d \in D$, the possibility that $P(d)$. The formula $\Diamond ! \exists x P(x)$, on the other hand, is also a conjecture, but it only highlights one possibility, namely, the possibility that someone has the property P.

The following proposition remarks some closure properties of the class of conjectures. We omit the utterly straightforward proof.

Proposition 6.3.25. For any assignment g and any formulas φ and ψ,

1. $\Diamond\varphi$ is a conjecture relative to g;
2. if one of φ and ψ is a conjecture relative to g, then so is $\varphi \vee \psi$;
3. if both φ and ψ are conjectures relative to g, then so is $\varphi \wedge \psi$;
4. if ψ is a conjecture relative to g, then so is $\varphi \to \psi$;
5. if φ is a conjecture relative to $g[x \mapsto d]$ for some $d \in D$, then $\exists x\varphi$ is a conjecture relative to g;
6. if φ is a conjecture relative to $g[x \mapsto d]$ for any $d \in D$, then $\forall x\varphi$ is a conjecture relative to g.

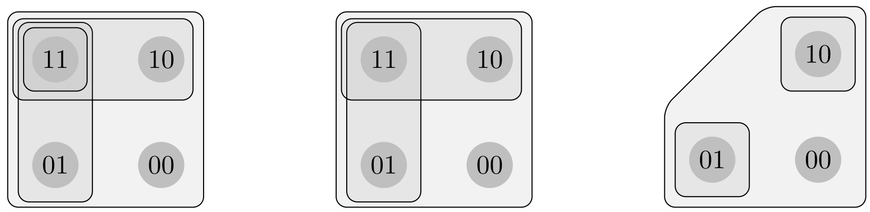
6.4 *Might* meets the logical constants

Since in the previous sections we have suggested a possible interpretation of the role of non-maximal possibilities and the operator $\Diamond$ in terms of 'might', it is worth spending at least a few words to discuss how well the formal behaviour of $\Diamond$ matches our intuitive expectations in some particular cases, in regards to its interaction with the other logical constants.

But before doing so, let us clear the table from a source of easy misapprehensions: when we say that $\Diamond$ may represent the effect of 'might', we are *not* talking about the *epistemic* 'might', meaning 'it is consistent with my information state that'. Inquisitive semantics is a very simple system which only models the information state of the *common ground* of a conversation relative to facts in the world. Neither the individual participants' knowledge, nor the common knowledge about the participants' knowledge is modelled in any way. Perhaps in the future inquisitive semantics may be extended with an epistemic component and will be able to cope with utterances concerning higher-order information, but until then there is no space for the modelling of the *epistemic* meaning of 'might'.

Instead, the 'might' discussed here is the one used to advance suggestions in conversation, to invite the other agents to consider certain possibilities, like in (1):

(1) They are late. They might be stuck in traffic.



(a) $[\![\Diamond p \wedge \Diamond q]\!]$ (b) $[\![\Diamond p \vee \Diamond q]\!]$ (c) $[\![\Diamond p \wedge \Diamond\neg p]\!] = [\![\Diamond p \vee \Diamond\neg p]\!]$

We call this the *suggestive*, or *highlighting* use of 'might'. These two functions of 'might' are distinct, although a connection between the two is probably present at a pragmatic level, in the sense that a suggestive use of might seems to imply that the corresponding epistemic might is true for the speaker, as well as the epistemic might of the negation of the suggestion: under normal circumstances, the hearer of (1) would conclude that the speaker does not know whether "they" are in fact stuck in traffic.

We start our test from the boolean connectives. A first thing that is immediate to see is that the formula $\neg\Diamond\varphi$ is always a contradiction. This is satisfactory, since it does not seem to be possible to deny a *might* statement in natural language: a sentence of the form "it might not p" it is invariably interpreted as suggesting the possibility that $\neg p$, and so it has its formal counterpart in $\Diamond\neg\varphi$.

Let us now consider the interaction with conjunction and disjunction. Figures 6.2(a) and 6.2(b) show the difference between the meanings of $\Diamond p \wedge \Diamond q$ and $\Diamond p \vee \Diamond q$: we see that both formulas suggest the possibility that p and the possibility that q —as we would expect— but the former also suggests the possibility that 'both', whereas the latter does not.

This may indeed be the desirable behaviour, as the following example pointed out by Anna Szabolcsi suggests. Suppose someone needs to have a text translated from Russian to French and is therefore looking for somebody who is a speaker of both languages. Then (2) would be perceived as a useful recommendation, whereas (3) would not.

(2) Mark might speak Russian and he might speak French.

(3) Mark might speak Russian or he might speak French.

In this context neither the suggestion that Mark speaks Russian nor the suggestion that Mark speaks French is of interest, but the suggestion that Mark speaks both *is*. Including the latter among the suggestions of $\Diamond p \wedge \Diamond q$, our system accounts for the different ways (2) and (3) are perceived.

On the other hand, Zimmermann (2000) has observed that (4) and (5) *are* equivalent.

(4) Mark might be in Paris or he might be in London.

(5) Mark might be in Paris and he might be in London.

In view of what we just saw, this may look in contrast with the predictions of our system, but it is not. For, being in London and being in Paris are obviously mutually exclusive propositions, and in (ordinary) conversations this is part of the common ground. Therefore, we should compute propositions not relative to the completely ignorant common ground consisting of all indices, but relative to the common ground from which the index 11 making both propositions true has been removed.

We have not discussed the generalization of possibility semantics and its notions to arbitrary common grounds, and we will not do it here either. This can be done in a straightforward way by restricting the clauses in the inductive definition of propositions to those indices that are actually there in the common ground. If we do so, we find out that indeed, relative to the common ground $\{10, 01, 00\}$, the formulas $\Diamond p \wedge \Diamond q$ and $\Diamond p \vee \Diamond q$ express the same proposition, namely $\{|\top|, |p|, |q|\}$, as depicted in figure 6.2(c).

Our system also associates to $\Diamond(p \wedge q)$ the expected proposal, namely $\{\mathcal{I}, |p \wedge q|\}$, and predicts the strong equivalence between $\Diamond\varphi \vee \Diamond\psi$ and $\Diamond(\varphi \vee \psi)$, which is also remarked by Zimmermann in the case of the above example and seems unproblematic.

So far so good. Now let us consider the interaction with the quantifiers. Both sentences $\exists x \Diamond P(x)$ and $\Diamond \exists x P(x)$ have the effect of suggesting the possibilities $P(d)$ for $d \in D$. Now, consider the following statement.

(6) Someone might have a map.

Apart for the trivial "nodding"-response, the ideal responses to (6) are clearly of the form "Yes, d has a map". Thus, the predictions of the systems seem to be correct.[1] On the other hand, other sentences of the same form seem to be better represented by the conjecture connected with the purely assertive component of the existential. For instance, (7) seems well-represented by the formula $\Diamond ! \exists x P(x)$, which simply suggests the possibility that $\exists x P(x)$.

(7) Someone might have stolen your bike.

Unfortunately, things are not as straight in the case of the universal quantifier. According to our system, the formula $\forall x \Diamond P(x)$ proposes not only the possibilities $|P(d)|$ for $d \in D$, but also all the intersections of such possibilities, including the possibility that $\forall x P(x)$. It is not clear to me whether this represents the meaning of any natural language statement. Consider the following two sentences.

[1] Someone might argue that a speaker who utters (6) need not believe that all the possibilities $P(d)$ are in fact possible; he might know, say, that Mark does not have a map. But remember what was said at the beginning of the section: inquisitive semantics only models the effect of utterances on the common ground of a conversation; it does not deal with the individual knowledge.
It may look suspicious that (6) should come out equivalent to the disjunction "Mark might have a map, or Elaine might have a map, or…", but it seems plausible that the difference between the two lies in the different conclusions that the hearer may draw about the knowledge state of the speaker, conclusions that are out of the scope of inquisitive semantics.

(8) Anyone might have a map.

(9) Everyone might have a map.

Statements like (9) seem to be used as equivalents of "It might be that everyone has a map", and therefore could simply be represented by the conjecture $\Diamond\forall x P(x)$, which simply highlights the possibility that $\forall x P(x)$. It is not so intuitively clear to me what possibilities (8) should suggest. The fact is that the 'might' in (8) seems to be essentially an epistemic one: for, the primary use of (8) is that of conveying information about one's own (lack of) information rather inviting the other agents to consider certain possibilities by advancing positive suggestions.

6.5 An assessment of possibility semantics

In the previous section we discussed the account of 'might' arising from the proposal of interpreting the formula $\chi \vee \top$ as "it might be that χ" if χ is an assertion. We saw that several phenomena relative to the potential of 'might' of highlighting possibilities are accounted for nicely, although there are important aspects of 'might' that cannot be represented in inquisitive semantics as it is.

However, it must be said that there are circumstances in which the interpretation of possibilities included in maximal possibilities as suggestions does not seem right. For instance, let us consider again the boundedness example: if we are in a common ground in which it is known that 4 is an upper bound to P, then the system correctly predicts that an utterance of the boundedness formula should be neither informative nor inquisitive, but it also predicts that it should advance the suggestions that 3, 2, 1, and 0 might be upper bounds for P, whereas our intuition is that it should simply be redundant.

Nonetheless, there is a crucial point that I want to make in favour of possibility semantics: even if the whole idea of interpreting possibilities included in maximal ones as suggestions should turn out to be altogether wrong, the system we have devised still allows for a satisfactory treatment of issues and information also in a first-order context, which is after all what we were originally after.

It extends both the propositional inquisitive semantics of chapter 2 and Groenendijk's partition semantics (Groenendijk, 1999) in a very natural way, it allows for the representation of a broad range of propositions and for the classification of formulas according to their effect in a dialogue, retains decent logical properties in virtue of its connection with the notion of support, and most importantly provides a notion of *resolutions*, which forms the basis for the study of notions that are crucial to the inquisitive programme, such as answerhood and compliance.

If it turns out that we really cannot make sense of possibilities that are strictly included in maximal possibilities, we can always choose to "filter" propositions, retaining only those possibilities that are *not* included in a maximal one. This is a weak version of the maximalization operation used to obtain meanings in chapter 2, and may be performed in two different ways: either *a posteriori*, in a single step, defining $\langle\!\langle\varphi\rangle\!\rangle = \mathsf{fil}([\![\varphi]\!])$, where fil has the effect of deleting those possibilities that are included in maximal ones; or in stages, removing the undesired possibilities inductively, with rules such as:

$$\langle\!\langle\varphi \vee \psi\rangle\!\rangle = \mathsf{fil}(\langle\!\langle\varphi \vee \psi\rangle\!\rangle \cup \langle\!\langle\varphi \vee \psi\rangle\!\rangle)$$

In the propositional case both approaches will simply give us back the maximalization semantics of chapter 2; on the other hand, in the first order case, we will still retain enough possibilities to represent the issues raised by any formula (which, as we saw in example 6.3.10 cannot be defined purely in terms of support). For instance, none of the possibilities proposed by the boundedness example would be filtered out, since the boundedness formula has no maximal possibility. At the same time, we will avoid many of the things that may be perceived as oddities in possibility semantics, for instance the fact exemplified above that formulas may still advances suggestions even when they are resolved.

6.6 Notes on first-order inquisitive logic

6.6.1 Definition and basic properties

In this section we will move the first steps towards an investigation of the logic that first-order inquisitive entailment gives rise to. For the sake of simplicity we shall make the assumption that our language is not equipped with equality and does not include function symbols.

Recall that, on a fixed domain D and relative to an assignment g into D, we say that a formula φ entails ψ in case $s, g \models \varphi$ implies $s, g \models \psi$ for any $D-$state s. We can then define a notion of "absolute" entailment as follows.

Definition 6.6.1 (First-order inquisitive entailment)**.** We say that a set of formulas Θ entails a formula φ, in symbols $\Theta \models_{\mathsf{InqQL}} \varphi$, if Θ entails φ on every domain relative to every assignment.

Definition 6.6.2 (Validity and logic)**.** We say that a first-order formula φ is *inquisitively valid* if $\models_{\mathsf{InqQL}} \varphi$, that is, if for any domain D, any $D-$state s and any assignment g into D we have $s, g \models \varphi$.

First-order inquisitive logic, denoted by InqQL, is simply the set of inquisitively valid formulas.

Like in the propositional case, inquisitive entailment amounts to classical entailment for assertions: in particular, by corollary 6.3.6 first-order inquisitive logic has the same disjunction-and-existential-free fragment as classical logic.

Also, like in the propositional case, for any formula φ we can characterize the declarative $!\varphi$ as the strongest assertion entailed by φ.

Obviously, a formula with free variables is inquisitively valid if and only if its universal closure is. Moreover, persistence implies that to determine validities we only have to look at the ignorant states $\mathcal{I}_D$ for all domains D.

Remark 6.6.3. A formula φ is inquisitively valid in case $\mathcal{I}_D, g \models \varphi$ for any domain D and assignment g into D.

As one might expect, the deduction theorem still holds.

Proposition 6.6.4 (Deduction theorem)**.** For any formulas $\theta_1, \ldots, \theta_n, \varphi$:

$$\theta_1, \ldots, \theta_n \models_{\mathsf{InqQL}} \varphi \iff \theta_1 \wedge \cdots \wedge \theta_n \to \varphi \in \mathsf{InqQL}$$

Proof. $\theta_1, \ldots, \theta_n \models_{\mathsf{InqQL}} \varphi$
$\iff$ for any domain D, any $s \in \mathcal{I}_D$ and any assignment g, if $s \models \theta_i$ for $1 \leq i \leq n$, then $s \models \varphi$
$\iff$ for any domain D, any $s \in \mathcal{I}_D$ and any assignment g, if $s \models \theta_1 \wedge \cdots \wedge \theta_n$ then $s \models \varphi$
$\iff$ for any domain D and any assignment g, $\mathcal{I}_D, g \models \theta_1 \wedge \cdots \wedge \theta_n \to \varphi$
$\iff$ $\theta_1 \wedge \cdots \wedge \theta_n \to \varphi \in \mathsf{InqQL}$ □

The logic InqQL is closed under modus ponens and under the natural rules for quantifiers: $\frac{\varphi \to \psi}{\varphi \to \forall x \psi}$ and $\frac{\psi \to \varphi}{\exists x \psi \to \varphi}$ where the variable x does not occur free in φ.

Like in the propositional case, replacing atoms by arbitrary formulas is not a sound operation in general: atomic formulas can only be soundly replaced by assertions, that is, by formulas for which the double negation law holds.

In the next section we are going to show that InqQL lies in between first-order intuitionistic logic IQL and first-order classical logic CQL.

6.6.2 $\mathsf{IQL} \subseteq \mathsf{InqQL} \subseteq \mathsf{CQL}$

We start from the observation that first-order inquisitive logic in included in classical logic. For, if $\varphi \notin \mathsf{CQL}$ there is a model M and an assignment g such that $M, g \not\models \varphi$; but then by the classical behaviour of singletons (proposition 6.1.6) we have $\{M\}, g \not\models \varphi$ and therefore $\varphi \notin \mathsf{InqQL}$. This shows the following fact.

Remark 6.6.5. $\mathsf{InqQL} \subseteq \mathsf{CQL}$

But there is something more to say about the connections of InqQL to CQL. For, corollary 6.3.7 guarantees that for any formula φ we have $\neg\neg\varphi = \{|\varphi|\}$, and therefore the following fact holds.

Proposition 6.6.6. For any first-order formula φ,

$$\varphi \in \mathsf{CQL} \iff \neg\neg\varphi \in \mathsf{InqQL}$$

In other words, Glivenko's theorem is true for first-order inquisitive logic, which is interesting since the same theorem does *not* hold for first-order intuitionistic logic.

Let us now turn to the connections with intuitionistic logic. We start by recalling the first-order analogue of the propositional Kripke models we used throughout the previous sections.

Definition 6.6.7. A first-order intuitionistic Kripke model is a pair $\mathcal{M} = (F, M)$ where $F = (W, \leq)$ is an intuitionistic Kripke frame and M is a function that associates to each point $w \in W$ a first-order model M_w for the language $\mathcal{L}$.[2] The map M must be *persistent*, that is, if $w \leq v$:

1. $D_w \subseteq D_v$ where D_w and D_v denote, respectively, the domains of M_w and M_v;
2. $R^{M_w} \subseteq R^{M_v}$ where R^{M_w} and R^{M_v} denote, respectively, the interpretation of a predicate symbol R in M_w and M_v.

Satisfaction of first-order formulas on an intuitionistic Kripke model $\mathcal{M}$ relative to an assignment is defined as follows.

Definition 6.6.8 (Kripke satisfaction)**.** For any first-order Kripke model $\mathcal{M}$, any point w and any assignment g into the model M_w:

1. if φ is atomic, $\mathcal{M}, w, g \Vdash \varphi$ iff $M_w, g \models \varphi$ classically;
2. the clauses for the Boolean connectives are the usual ones;
3. $\mathcal{M}, w, g \Vdash \exists x\varphi$ if there is a $d \in D_w$ such that $\mathcal{M}, w, g[x \mapsto d] \Vdash \varphi$;
4. $\mathcal{M}, w, g \Vdash \forall x\varphi$ if for all $v \geq w$ and all $d \in D_v$ it is $\mathcal{M}, v, g[x \mapsto d] \Vdash \varphi$.

We then say that a formula is valid on a model $\mathcal{M}$ if it is satisfied everywhere in $\mathcal{M}$ under any assignment. Of course, the crucial feature of such models is that they provide a sound and complete semantics for first-order intuitionistic logic.

Theorem 6.6.9. For any first-order formula φ, $\varphi \in \mathsf{IQL}$ if and only if φ is valid on any first-order intuitionistic Kripke model.

Now, in section 2.2 we saw that in the propositional case, inquisitive support amounts to satisfaction on a particular intuitionistic Kripke model. Is this the same for first-order inquisitive support? The answer is *yes*, although now we will have one model for any particular choice of the underlying domain D.

Definition 6.6.10 (Kripke models for inquisitive logic)**.** Given a set D, the *Kripke model for inquisitive semantics* over D is the model ${\mathcal{M}_I}^D$ defined as follows:

1. the underlying frame is simply ${\mathcal{F}_I}^D = (\mathcal{I}_D - \emptyset, \supseteq)$;
2. for any state s, the model M_s consists of the domain D with the interpretation of the predicate symbols given by: $R^{M_s} := \bigcap_{N \in s} R^N$.

[2] If we also had function symbols and equality, the former should be interpreted as *partial* functions with a persistence condition, and the latter as a congruence relation that gets less refined as we go up the accessibility relation.

It is clear that $(\mathcal{I}_D - \emptyset, \supseteq)$ is a partial order. Let us check that the persistence conditions are satisfied as well. The domains and the interpretation of the function symbols are constant, so they are obviously persistent. Moreover, consider a relation symbol R: if $s \supseteq t$ we have $R^{M_s} = \bigcap_{M \in s} R^M \subseteq \bigcap_{M \in t} R^M = R^{M_t}$.

Thus, ${\mathcal{M}_I}^D$ is indeed a first-order intuitionistic Kripke model. The next proposition states that inquisitive support on D-states amounts to Kripke satisfaction on ${\mathcal{M}_I}^D$.

Proposition 6.6.11. For any formula φ, any non-empty D-state s and any assignment g,

$$s, g \models \varphi \iff {\mathcal{M}_I}^D, s, g \Vdash \varphi$$

Proof. Proceed by induction on the formula φ.

1. Basic case. Consider an atomic formula $R(x_1, \ldots, x_n)$ and an assignment g, and for each i let $d_i = g(x_i)$.

   We have: $s, g \models R(x_1, \ldots, x_n)$
   $\iff M, g \models R(x_1, \ldots, x_n)$ for all models $M \in s$
   $\iff \langle d_1, \ldots, d_n \rangle \in R^M$ for all $M \in s \iff \langle d_1, \ldots, d_n \rangle \in \bigcap_{M \in s} R^M$
   $\iff \langle d_1, \ldots, d_n \rangle \in R^{M_s} \iff {\mathcal{M}_I}^D, s, g \Vdash R(x_1, \ldots, x_n)$.

2. The induction steps for the Boolean connectives and the existential quantifier are straightforward.

3. Universal quantifier. If ${\mathcal{M}_I}^D, s, g \Vdash \forall x \varphi$, then in particular, for any $d \in D$ we must have ${\mathcal{M}_I}^D, s, g[x \mapsto d] \Vdash \varphi$, whence by induction hypothesis $s, g[x \mapsto d] \models \varphi$: thus, $s, g \models \forall x \varphi$. Conversely, if $s, g \models \forall x \varphi$, then for all $d \in D$ we have $s, g[x \mapsto d] \models \varphi$. Now consider a non-empty substate $t \subseteq s$ and an element d in the associated domain, which by definition is simply D: by persistence, since $s, g[x \mapsto d] \models \varphi$ also $t, g[x \mapsto d] \models \varphi$. Hence, ${\mathcal{M}_I}^D, s, g \Vdash \forall x \varphi$.

□

As a corollary we obtain the above mentioned fact that first-order inquisitive logic includes intuitionistic logic.

Corollary 6.6.12. $\mathsf{IQL} \subseteq \mathsf{InqQL}$

Proof. If $\varphi \notin \mathsf{InqQL}$, there is a domain D, a non-empty D-state s and an assignment g such that $s, g \not\models \varphi$. Then by the previous proposition ${\mathcal{M}_I}^D, s, g \not\Vdash \varphi$, and since ${\mathcal{M}_I}^D$ is an intuitionistic Kripke model, $\varphi \notin \mathsf{IQL}$. □

Clearly, both inclusions are proper. For, instance, the law of excluded middle does not hold in InqQL, while several examples of inquisitively valid formulas that are not valid in intuitionistic logic are provided by the following remark. The straightforward proofs have been omitted.

Remark 6.6.13. The following formulas are in $\mathsf{InqQL} - \mathsf{IQL}$.

1. Atomic double negation: $\neg\neg R(x_1, \ldots, x_n) \to R(x_1, \ldots, x_n)$ for any atomic formula $R(x_1, \ldots, x_n)$.

2. Kreisel-Putnam scheme (and in fact any instance of a propositional scheme which is valid in Medvedev logic):

$$(\neg\chi \to \varphi \vee \psi) \to (\neg\chi \to \varphi) \vee (\neg\chi \to \varphi)$$

3. The infinite equivalent of the Kreisel-Putnam scheme:

$$(\neg\chi \to \exists x\varphi(x)) \to \exists x(\neg\chi \to \varphi(x))$$

   where x is not free in χ.

4. The following formula, which characterizes the models with constant domains:

$$\forall x(\chi \vee \varphi(x)) \to \chi \vee \forall x\varphi(x)$$

   where x is not free in χ.

5. Finally, the axiom scheme that allows the extension of Glivenko's theorem to the first-order setting:

$$\forall x\neg\neg\varphi \to \neg\neg\forall x\varphi$$

It is dubious whether augmenting a Hilbert-style system for IQL with these five principles would yield a complete axiomatization of InqQL; this is probably not the case.

There is, however, at least one result we obtained in the propositional case that can be reproduced in the first-order setting, namely the correspondence theorem stating that InqL is the logic of negative saturated models, with the extra restriction of constant domains.

6.6.3 Correspondence theorem

Definition 6.6.14 (Negative models)**.** We say that a first-order intuitionistic Kripke model $\mathcal{M}$ is *negative* in case the formula $\neg\neg R(x_1, \ldots, x_n) \to R(x_1, \ldots, x_n)$ is true everywhere on $\mathcal{M}$ for any *atomic* formula $R(x_1, \ldots, x_n)$.

This amounts to requiring that for any relation symbol R in the language and any non-terminal point w of the model, $R^{M_w} = \bigcap_{v>w} R^{M_v}$. In case any point has access to an endpoint, this can be reformulated as $R^{M_w} = \bigcap_{e\in E_w} R^{M_e}$ where E_w denotes the set of endpoints of w.

This shows that if a model is negative and has enough endpoints, then the first-order model attached to a point w is determined by the models attached to the endpoints of w.

Notation. If K is a class of first-order Kripke models, we denote by $\mathbf{N}K$ the class of negative models in K and by $\mathbf{C}K$ the class of models in K with constant domains. Similarly, if K is a class of Kripke frames, we denote by $\mathbf{N}K$ the class of negative models over frames in K and by $\mathbf{C}K$ the class of models with constant domains over frames in K.

Thus, **NCSat** denotes the class of negative saturated models with constant domains (saturated frames were defined in 3.2.17). We shall prove the following fact.

Theorem 6.6.15 (Correspondence theorem)**.** InqQL=$\mathsf{Log}(\mathbf{NCSat})$.

One direction of the theorem is easy. For, if $\varphi \notin \mathsf{InqQL}$, then there must be a domain D, a $D-$state s and an assignment g such that $s, g \not\models \varphi$; then by proposition 6.6.11 we have $\mathcal{M}_I{}^D, s, g \not\Vdash \varphi$. But it is immediate to check that $\mathcal{M}_I{}^D$ is indeed a negative saturated model with constant domains (the proof of saturation is the same as in the propositional case), so $\varphi \notin \mathsf{Log}(\mathbf{NCSat})$. This proves $\mathsf{Log}(\mathbf{NCSat}) \subseteq \mathsf{InqQL}$. The converse direction relies on an analogue of lemma 3.2.19 stating that any negative saturated model can be p-morphically mapped into the Kripke model for inquisitive semantics.
In order to even *state* such an analogue we need of course to specify a first-order analogue of the notion of p-morphism. It is not hard to imagine what this should be.

Definition 6.6.16 (p-morphisms)**.** Call a map $\eta : \mathcal{M} \to \mathcal{M}'$ between first-order Kripke models a *p-morphism* in case it is a frame p-morphism and, in addition, for any point w in $\mathcal{M}$ the models M_w and $M'_{\eta(w)}$ are isomorphic.

It is straightforward to check that the satisfaction of sentences is invariant under such morphisms; that is, if η is as above, for any sentence φ and any point w we have $\mathcal{M}, w \Vdash \varphi \iff \mathcal{M}', \eta(w) \Vdash \varphi$.

Lemma 6.6.17. For any $\mathcal{M} \in \mathbf{NCSat}$ there is a p-morphism $\eta : \mathcal{M} \to \mathcal{M}_I{}^D$, where D is the particular domain on which the models associated to points in $\mathcal{M}$ are based on.

Proof. Given $\mathcal{M} \in \mathbf{NCSat}$, define our candidate p-morphism as follows. For any point w in $\mathcal{M}$, $\eta(w) = \{M_e \,|\, e \in E_w\}$ where E_w denotes the set of endpoints of w.

Since all first-order models attached to points in $\mathcal{M}$ are based on the same domain D, $\eta(w)$ is a $D-$state. Moreover, $E_w \neq \emptyset$ by the E-saturation condition, so in fact $\eta(w)$ is a non-empty $D-$state, i.e. a point in the Kripke model $\mathcal{M}_I{}^D$. This shows that η is at least a well-defined map into $\mathcal{M}_I{}^D$.

It is evident that η satisfies the forth condition. For the back condition, argue like in the propositional case exploiting the I-saturation condition.

It remains to show that for each point w the first-order models M_w and $M_{\eta(w)}$ are isomorphic; in fact, we are going to show that for each relation symbol R we have $R^{M_w} = R^{M_{\eta(w)}}$, so that the identity itself is an isomorphism between M_w and $M_{\eta(w)}$.

Fix a point w and a relation symbol R. Since $\mathcal{M}$ is negative and any point in $\mathcal{M}$ has access to an endpoint, $R^{M_w} = \bigcap_{e \in E_w} R^{M_e}$. Then by definition of the Kripke model ${\mathcal{M}_I}^D$ and of the map η we have:

$$R^{M_{\eta(w)}} = \bigcap\nolimits_{M \in \eta(w)} R^M = \bigcap\nolimits_{e \in E_w} R^{M_e} = R^{M_w}$$

This completes the proof. □

The missing direction of theorem 6.6.15 follows now speedily from the lemma.

Proof of theorem 6.6.15, concluded. We have shown above that $\mathsf{Log}(\mathbf{NCSat}) \subseteq \mathsf{InqQL}$. Now consider the other direction: we need just consider sentences, since the validity of a formula always comes down to the validity of its universal closure. So, consider a sentence φ and suppose $\varphi \notin \mathsf{Log}(\mathbf{NCSat})$: this means that there is a negative saturated Kripke model $\mathcal{M}$ with constant domains and a point w such that $\mathcal{M}, w \not\Vdash \varphi$. Now let D denote the domain of the first-order models associated to points in $\mathcal{M}$: the previous lemma gives us a p-morphism $\eta : \mathcal{M} \to {\mathcal{M}_I}^D$. By the invariance of satisfaction under p-morphisms we have ${\mathcal{M}_I}^D, \eta(w) \not\Vdash \varphi$, whence according to proposition 6.6.11 we have $\eta(w) \not\models \varphi$ and therefore $\varphi \notin \mathsf{InqQL}$. □

In chapter 3 we remarked that the analogue of the previous theorem, stating that $\mathsf{InqL} = \mathsf{Log}(\mathbf{nSAT}) = \mathsf{Log}(\mathbf{SAT})^n$, provided a viable path to a sound and complete axiomatization of InqL. The argument was the following: suppose we can axiomatize an intermediate logic Λ for which $\Lambda = \mathsf{Log}(K)$ for some class K of Kripke frames with $F_I \in K \subseteq \mathbf{SAT}$: then we have $\mathsf{InqL} = \mathsf{Log}(\{M_I\}) \supseteq \mathsf{Log}(\mathbf{n}K) \supseteq \mathsf{Log}(\mathbf{nSAT}) = \mathsf{InqL}$, and therefore $\mathsf{InqL} = \mathsf{Log}(\mathbf{n}K) = \mathsf{Log}(K)^n = \Lambda^n$, which shows that InqL is axiomatized by a system obtained by adding the principle of atomic double negation to the axioms of Λ.

Obviously, in the predicate case we can define negative variants in the usual way. Arguing along the lines of the proof of proposition 3.2.11, one can easily show that for any class K of Kripke frames, $\mathsf{Log}(\mathbf{N}K) = \mathsf{Log}(K)^n$ and moreover also $\mathsf{Log}(\mathbf{NC}K) = \mathsf{Log}(\mathbf{C}K)^n$.

Corollary 6.6.18. If K is a class of Kripke frames with $\{{\mathcal{F}_I}^D \mid D \text{ a set}\} \subseteq K \subseteq \mathbf{SAT}$, then $\mathsf{InqQL} = \mathsf{Log}(\mathbf{C}K)^n$.

Proof. If $\{{\mathcal{F}_I}^D \mid D \text{ a set}\} \subseteq K \subseteq \mathbf{SAT}$, then $\{{\mathcal{M}_I}^D \mid D \text{ a set}\} \subseteq \mathbf{NC}K \subseteq \mathbf{NCSat}$, so $\mathsf{InqQL} = \mathsf{Log}(\{{\mathcal{M}_I}^D \mid D \text{ a set}\}) \supseteq \mathsf{Log}(\mathbf{NC}K) \supseteq \mathsf{Log}(\mathbf{NCSat}) = \mathsf{InqQL}$. Hence, $\mathsf{InqQL} = \mathsf{Log}(\mathbf{NC}K) = \mathsf{Log}(\mathbf{C}K)^n$. □

With this corollary we have reduced the problem of axiomatizing InqQL to the problem of axiomatizing any logic of the shape $\mathsf{Log}(\mathbf{C}K)$ for some class K of Kripke frames with $\{{\mathcal{F}_I}^D \mid D \text{ a set}\} \subseteq K \subseteq \mathbf{SAT}$.

Chapter 7

Conclusions

This thesis has been concerned with the development of inquisitive semantics for both a propositional and a first-order language, and with the investigation of the logical systems they give rise to.

In the first place, we discussed the features of the system arising from the semantics proposed by (Groenendijk, 2008a) and (Ciardelli, 2008), explored the associated logic and its connections with intermediate logics and established a whole range of sound and complete axiomatizations; these are obtained by expanding certain intermediate logics, among which the Kreisel-Putnam and Medvedev logics, with the double negation axiom for atoms. We showed that the schematic fragment of inquisitive logic coincides with Medvedev's logic of finite problems, thus establishing interesting connections between the latter and other well-understood intermediate logics: in the first-place, ML is the set of schematic validities of a recursively axiomatized derivation system, obtained (for instance) by expanding the Kreisel-Putnam logic with atomic double negation axioms; in the second place, a formula φ is provable in Medvedev's logic if and only if any instance of it obtained by replacing an atom with a disjunction of negated atoms is provable in the Kreisel-Putnam logic (or indeed in any logic within a particular range).

These results also prompted us to undertake a more general investigation of intermediate logics whose atoms satisfy the double negation law.

Furthermore, we showed how the original 'pair' version of inquisitive semantics can be understood as one of a hierarchy of specializations of the 'generalized' semantics we discussed, and argued in favour of the generalized system.

Finally, we turned to the task of extending inquisitive semantics to a first-order language and found that a straightforward generalization of our propositional approach was not viable due to the absence of certain maximal states. In order to overcome this difficulty, we proposed a variant of the semantics, which we called inquisitive *possibility* semantics, based on an inductive definition of possibilities.

We examined the resulting system, arguing that it retains most of the properties of the semantics discussed in the previous chapters, including the logic, and we proposed a possible way to interpret the additional aspects of meaning that appear in the new semantics. We discussed the distinction between entailment and *strong* entailment and gave a sound and complete axiomatization of the latter notion as well. We showed that possibility semantics can be extended naturally to the predicate case, tested the predictions of the resulting system and found them satisfactory, especially in regard to the treatment of issues and information, and we saw that the system comprehends Groenendijk's logic of interrogation as a special case. We concluded sketching some features of the associated predicate logic.

The semantics we have discussed are new, in fact *completely* new in the case of possibility semantics and its first-order counterpart. I hope to have managed to provide some evidence of their great potential for linguistic applications: in a very simple system and without any *ad-hoc* arrangement, we can deal with phenomena such as polar, conditional and *who* questions, inquisitive usage of indefinites and disjunction, perhaps even *might* statements, all of this in symbiosis with the classical treatment of information. For obvious reasons, in this thesis we have limited ourselves to remarking that these phenomena *can* be modelled: of course, a great deal of work remains to be done in order to understand what account each of them is given in inquisitive semantics.

Aspects that may be worth particular consideration are the notions of *answerhood* and *compliance* that the semantics gives rise to, as well as the type of pragmatic inferences it justifies. Also, the role of suggestive possibilities (if any) and their relations to natural language constructions such as *might* and *perhaps* has to be clarified.

From the logical point of view, natural directions of research are a more in-depth study of inquisitive logic and strong entailment in the first-order case, possibly leading up to a syntactic characterization.

Beyond the borders of inquisitive semantics, a further possible stream of research along the lines of chapter 5 would be to study the behaviour of intermediate logics with atoms satisfying special classical properties (say, the Scott formula or the Gödel-Dummett formula) or perhaps even *arbitrary* properties. The resulting objects would be weak logics (weak *intermediate* logics in the case of classical properties) and may therefore be studied by means of constructions analogous to those devised in chapters 3 and 5 for the particular case of the double negation property.

Finally, we hope that the connections established here between Medvedev's logic and decidable logics such as ND and KP may serve as a useful tool to cast some light on this ever-mysterious intermediate logic and on the long-standing issue of its decidability.

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