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Questions as information types

Ivano Ciardelli

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Abstract This paper argues that there is an important role to be played by questions in logic, both semantically and proof-theoretically. Semantically, we show that, by generalizing the classical notion of entailment to questions, we can capture not only the standard relation of logical consequence, which concerns specific pieces of information, but also the relation of logical *dependency*, which concerns information *types*. Proof-theoretically, we show that questions may be used in inferences as placeholders for generic information of a given type; by manipulating such placeholders, we may construct formal proofs of dependencies. Finally, we show that such proofs have a specific kind of constructive content: they do not just witness the existence of a certain dependency, but actually encode a method for computing it.

Keywords Questions · Dependency · Inquisitive logic · Proofs-as-programs

1 Introduction

1.1 A motivating example

Suppose a certain disease may give rise to two symptoms, S_1 and S_2, the latter much more distressing than the former. Suppose the disease may be countered by means of a certain treatment, which however carries some associated risk. A hospital has the following protocol for dealing with the disease: if a patient presents symptom S_2, the treatment is always prescribed. If the patient only presents symptom S_1, however, the treatment is prescribed just in case the patient is in good physical condition; if not, the risks associated with the treatment outweigh the benefits, and the treatment is not administered.

I.A. Ciardelli
Science Park 107, Amsterdam
Tel.: +31-20-5256011
E-mail: sobekal@hotmail.com

Given the hospital's protocol, whether or not the treatment is prescribed for a patient is determined by two things: (i) which symptoms the patient presents and (ii) whether the patient is in good physical condition. This means that, in the given context, a certain relation holds between the following questions:

μ_1. What the patient's symptoms are
μ_2. Whether the patient is in good physical condition
ν . Whether the treatment is prescribed

This relation amounts to the following: in the given context, settling the questions μ_1 and μ_2 implies settling the question ν. We will say that, the question ν is *determined* by the questions μ_1 and μ_2 in the given context, and we will refer to this relation as a *dependency*.[1]

This relation may also be viewed as connecting three different types of information: given the hospital's protocol, complete information about a patient's symptoms, combined with information about whether the patient is in good condition, yields information about whether the treatment is prescribed. Using the questions as labels, we may say that information of type μ_1, together with information of type μ_2, yields information of type ν.

1.2 The relevance of dependency

Dependencies are quite ubiquitous, both in ordinary contexts, such as the one of our example, and in specific scientific domains. In this section we mention three areas where this notion plays a role, although undoubtedly many others can be found.

Natural sciences. Much of the enterprise of natural sciences such as physics and chemistry consists in finding out what dependencies hold in nature: what are those factors that determine the trajectory of a planet, the temperature of a gas, or the speed of a certain chemical reaction?

One of the earliest achievements of modern science was the discovery that, absent air resistance, the time that a body dropped near the Earth surface employs to reach the ground is completely determined by the height from which it is dropped; another early realization of the modern theory of mechanics is that, given a flat surface, the distance at which a cannonball will land is completely determined by its initial velocity. Such relations are all cases of dependency in our sense: one question (say, what the initial velocity is) completely determines another question (say, how far the cannonball will land).

Indeed, the epistemic value of a scientific theory, such as classical mechanics or thermodynamics, lies at least to a large extent in its ability to establish such

[1] This use of the word *dependency* is quite different from the ordinary sense of the word. If we have a dependency of a question ν on another question μ, then this means that ν is *completely*, and not only partially, determined by μ. *Determinacy* would probably be a better term for this notion, but we stick to *dependency* for the sake of consistency with the existing literature.

dependencies, which is often referred to as the theory's *predictive power*. Our perspective allows us to make this very precise: we can say that a theory Γ is predictive of a question ν given questions $\mu_1, \ldots, \mu_n$ in case within the context of Γ, ν is determined by $\mu_1, \ldots, \mu_n$. Thus, e.g., classical mechanics can be said to be predictive of a body's position at a time t given (i) the body's position and velocity at a different time t_0, (ii) the body's mass and (iii) the force field in which the body moves.

Linguistics. One of the goals of the theory of pragmatics is to understand when a certain sequence of conversational moves forms a coherent dialogue, and why. A crucial part of this task is to characterize what sentences count as acceptable replies to a question in a certain context. Now consider the following exchange:

(1) Alice: Where can I find you tomorrow?
Bob: If it is sunny I'll be at the park; if not, I'll be at home.

In this dialogue, Bob's reply sounds as informative a response as Alice can possibly hope for. However, strictly speaking, it does not *resolve* Alice's question, since it does not provide a specific place where Bob can be found. Rather, what Bob's reply does is to establish a *dependency* of Alice's question on another question, the question whether it will be sunny. This illustrates the fact that in some cases, the optimal response to a given question may in fact take the form of a dependency on another question.

Databases. A database is a relation, i.e., a collection of vectors of a given size. A vector in a database is called an *entry*, and its coordinates are called the entry's *attributes*. E.g., the database of a university may contain one entry for each student, and the attributes may be *student ID*, *last name*, *program*, etc.

The traditional role of dependency in database theory is in the specification of constraints that the database should satisfy. Such constraints often take the form of dependencies. E.g., as a university we want an ID number to completely identify a student: this means that that the attribute *student ID* should completely determine the value of all other attributes in the database, such as *first name*, *program*, etc. Since it is important to verify that such a constraint is indeed satisfied by a database, reasoning about dependencies is a topic that has received much attention in the database community.

A related domain where dependency plays a role is *query answering*. Queries are essentially just questions in a specific formal language. When a query is asked by a user, a program accesses the database, computes an answer, and returns it to the user. However, databases are typically large, and consulting them is costly: thus, it is often useful to store the answers to particular queries after these have been answered. Such stored answers are called *views*. Ideally, when a query is asked, one would like to compute an answer just based on the available views, without having to reconsult the database. However, this is only possible if the new query is in fact *determined* by the views, i.e., if a certain dependency relation holds.

1.3 Aim and structure of the paper

The first purpose of this paper is to demonstrate that dependency is not an exotic logical notion, but a facet of the fundamental logical notion of entailment, once this is extended to cover not only statements, but also questions. As such, it can be investigated in an insightful way by means of the standard notions and tools of logic, provided logic is extended to encompass questions.

The second purpose of the paper is to show that questions have an interesting role to play in inferences. When occurring in a logical proof, a question plays the role of a placeholder, standing for an arbitrary piece of information of a certain type. For instance, the question *what the patient's symptoms are* stands for some complete specification of the patient's symptoms. By using questions, we can thus manipulate generic information, and this makes it possible to provide simple formal proofs of dependencies.

Finally, the third purpose of the paper is to show that such proofs admit a constructive interpretation, similar to the proofs-as-programs interpretation of intuitionistic logic: they do not just witness the existence of a dependency, but actually encode a method for computing the dependency, i.e., a method for turning information of the type described by the assumptions into information of the type described by the conclusion.

The paper is structured as follows: Section 2 discusses how questions can be brought within the scope of logic by moving from a truth-conditional semantics to an information-based semantics, and how dependency emerges as a facet of entailment in this generalized setting. Section 3 illustrates these ideas by means of a concrete formal system which extends classical propositional logic with questions. Section 4 deals with the role played by questions in proofs and brings out the constructive content of proofs involving questions. Section 5 situates the present contribution in a broader context, comparing it to other logical approaches to questions and dependency. Finally, Section 6 summarizes the main points of the paper and outlines directions for future work.

2 Entailment in the realm of questions

2.1 From truth-conditions to support-conditions

Traditionally, logic is concerned with relations between *statements*, that is, between sentences that may be seen as describing a state of affairs. Classical logic arises from the default assumption that the meaning of a statement lies in its truth-conditions, that is, in the conditions that a state of affairs must satisfy in order for the statement to be true.

We will refer to the formal representation of a (complete) state of affairs as a *possible world*, and we will denote the set of possible worlds by ω.[2] Thus,

[2] The exact nature of possible worlds depends on the specific logical framework. Usually, a possible world may be identified with a model for the language at stake. In *intensional logics*, which aim at representing a whole variety of states of affairs in a single model,

in the truth-conditional approach, semantics consists in the specification of a relation $w \models \alpha$ between possible worlds w and statements α, which holds in case α is true in the state of affairs described by w. The central notion of logic, the relation of *entailment*, can then be defined as preservation of truth: α entails β in case β is true whenever α is.

$$\alpha \models \beta \iff \text{for all } w \in \omega : \; w \models \alpha \text{ implies } w \models \beta$$

This is, in a nutshell, the usual way to make sense of the fundamental notion of entailment in classical logic. From this perspective, questions seem to have no place in logic: after all, it is not even clear what it should mean for a question to be *true* or *false* in a certain state of affairs. Since entailment is defined in terms of truth-conditions, it is also not clear what it would mean for a question to occur as an assumption or as a conclusion of an entailment relation.

However, it is possible to give an alternative semantic foundation for classical logic, which starts out from a more information-oriented perspective. Rather than taking the meaning of a statement α to be given by laying out in which circumstances α is true, we may take it to be given by laying out *what information it takes to settle that* α. In this perspective, α is evaluated not with respect to states of affairs, but instead with respect to pieces/bodies of information, whose formal counterpart we will call *information states*.[3]

A simple and perspicuous way of modeling an information state, which goes back at least to Hintikka (1962), and which is widely adopted both in logic and in linguistics, is to identify it with a set s of possible worlds, namely, those worlds that match (i.e., are compatible with) the available information. In other words, if s is a set of possible worlds, then s encodes the information that the actual state of affairs corresponds to one of the possible worlds in s. If $t \subseteq s$, this means that in t, at least as much information as in s is available, and possibly more. We say that t is an *enhancement* of s or also that t *entails* s.

In the informational approach that we will explore, semantics will thus be given by a relation $s \models \alpha$, called *support*, between information states s and statements α, which holds in case α is *settled* in s. This semantic perspective brings along a corresponding notion of entailment as preservation of support: α entails β in case settling that α implies settling that β.

$$\alpha \models \beta \iff \text{for all } s \subseteq \omega : \; s \models \alpha \text{ implies } s \models \beta$$

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possible worlds are internalized as particular entities within the model. We leave this notion unspecified in this section, since we want to abstract away from the details of a specific logical framework and look at the general template instead.

[3] Information-oriented semantics have been considered in the literature, especially as a starting point for non-classical logics (e.g., Beth, 1956; Kripke, 1965; Veltman, 1981), but sometimes also as alternative foundations for classical logics (e.g., Fine, 1975; Humberstone, 1981; van Benthem, 1986; Holliday, 2014). As far as the treatment of classical logic is concerned, our system will be very similar to the ones in the latter tradition, modulo a basic difference in the modeling of information states. To the best of my knowledge, however, no previous attempt has been made to use such a semantic foundation to bring questions into play in logic. Within the context of inquisitive semantics, the move from worlds to information states as points of evaluation is also discussed in some detail in Groenendijk (2011), which is a source of inspiration for the present paper.

Now, we regard a statement α as being settled in s just in case it follows from the information in s that α is true, i.e., in case s is only compatible with worlds in which α is true. But this means that s settles α iff all the worlds in s are worlds where α is true. Let us write $|\alpha|$ for the set of worlds where α is true, that is, $|\alpha| = \{w \in \omega \mid w \models \alpha\}$. Then, for all information states s we have:

$$s \models \alpha \iff s \subseteq |\alpha| \tag{1}$$

Thus, the support conditions for a statement are completely determined by its truth-conditions. On the other hand, if we consider this connection in the special case that s is a singleton $\{w\}$, we find that it also implies that, conversely, truth-conditions are determined by support conditions.

$$w \models \alpha \iff w \in |\alpha| \iff \{w\} \subseteq |\alpha| \iff \{w\} \models \alpha \tag{2}$$

Intuitively, what this says is that α is true at a world w just in case the information that w is the actual world establishes that α.

These connections show that, for statements, the truth-conditional approach and the support-conditional approach are two sides of the same coin: support conditions and truth conditions are interdefinable.

What is more, the truth-conditional notion of entailment and the support-conditional one coincide. To see this, suppose α truth-conditionally entails β, and suppose $s \models \alpha$: this means that $s \subseteq |\alpha|$. Since α truth-conditionally entails β, we have $|\alpha| \subseteq |\beta|$, and so also $s \subseteq |\beta|$. But then, we also have that $s \models \beta$. This shows that α entails β in the support-conditional sense.

Conversely, suppose α entails β in the support-conditional sense, and suppose $w \models \alpha$. Then, $\{w\}$ is a state which supports α. Since α entails β in the support-conditional sense, $\{w\}$ must also support β, which means that we must have $w \models \beta$. This shows that α entails β in the truth-conditional sense.

What all this means, in short, is that support semantics does not give rise to a new logic of its own, but instead provides an alternative, informational semantic foundation for classical logic.

2.2 Questions enter the stage

While truth-conditional semantics and support semantics are equivalent as far as statements are concerned, support semantics has an advantage which is not obvious at first: unlike truth-conditional semantics, it naturally accommodates not only statements, but also *questions*. For, while it is not clear what it means for a question to be true or false in a certain state of affairs, there is a natural sense in which a question can be said to be *settled* in an information state s: namely, when it is completely resolved by the information available in s.

For a concrete example, consider one of the questions in our example, the question μ_1 of what symptoms, out of S_1 and S_2, the patient presents. An information state s settles this question in case either (i) s settles that the patient presents neither symptom, or (ii) s settles that the patient presents only S_1, or (iii) s settles that the patient presents only S_2, or finally, (iv) s settles that the patient presents both symptoms. This means that μ_1 is settled in a state s just in case s is included in one of the following four states:

- $a_\emptyset \;= \{w \in \omega \,|\, \text{patient has no symptoms in } w\}$
- $a_1 \;= \{w \in \omega \,|\, \text{patient has only symptom } S_1 \text{ in } w\}$
- $a_2 \;= \{w \in \omega \,|\, \text{patient has only symptom } S_2 \text{ in } w\}$
- $a_{12} = \{w \in \omega \,|\, \text{patient has both symptoms in } w\}$

It is worth pointing out that not only are support conditions naturally defined for questions: there are also good reasons to regard them as a good candidate for the role of question meaning. For, questions are used primarily, though not uniquely, in order to specify requests for information: it is therefore natural to expect that to know the meaning of a question is to know what information is requested by asking it, that is, what information state has to be brought about in order for the question to be settled. That is precisely what is encapsulated into the question's support conditions.

2.3 Pieces and types of information

In truth-conditional semantics, the meaning of a sentence φ is encoded by its *truth-set*, i.e., by the set $|\varphi| = \{w \in \omega \,|\, w \models \varphi\}$ of worlds at which φ is true. Similarly, in support semantics, the meaning of a sentence φ is encoded by its *support-set*, that is, the set $[\varphi] = \{s \subseteq \omega \,|\, s \models \varphi\}$ of states which support φ.

The support-set of a sentence is a set of information states of a special form. For, suppose an information state s settles a sentence φ: then, any enhancement t of s will also settle φ. That is, the relation of support is *persistent*:[4]

Persistency: if $t \subseteq s$, $s \models \varphi$ implies $t \models \varphi$

This implies that the support-set $[\varphi]$ of a sentence is always *downward closed*, that is, if it contains a state s, it also contains all enhancements $t \subseteq s$.

Downward closure: if $t \subseteq s$, $s \in [\varphi]$ implies $t \in [\varphi]$

It will be useful to introduce a notion of *downward closure* of a given set T of information states, defined as follows:

$$T^{\downarrow} = \{s \subseteq \omega \,|\, s \subseteq t \text{ for some } t \in T\}$$

Clearly, $T^{\downarrow}$ is a downward closed set; in fact, $T^{\downarrow}$ is always the smallest downward closed set which contains T. We say that the set of states $T^{\downarrow}$ is generated by T, or that T is a *generator* for $T^{\downarrow}$.

Now, the support set of a *statement* has another important feature besides downward closure. For, Relation 1 between the truth-conditions of a statement

[4] At this stage, this may be taken as a stipulation about what it means for a sentence to be *settled*: a sentence is settled by virtue of certain information being available in a state; if $t \subseteq s$, all the information which is available in s is also available in t, and so, t cannot fail to support a sentence if s does. Given a specific logical system based on support semantics, such as the one described in Section 3, this property may then be proved as a fact.

and its truth-conditions implies that $s \in [\alpha] \iff s \subseteq |\alpha|$. That is, we have the following relation:

$$[\alpha] = \{|\alpha|\}^{\downarrow}$$

This shows that the support-set of a statement is always generated by a single state, namely, the truth-set $|\alpha|$.[5] We may regard $|\alpha|$ as a piece of information, namely, the information that α is true. Thus, we may regard a statement α as describing a specific piece of information. To say that α is settled in a state is simply to say that this piece of information is available in s.

This is not the case for questions. For instance, consider again the the question μ_1 of what symptoms the patient presents, in the context of our example. We saw above that a state s supports μ_1 if it is included in one of the following four states:

- $a_\emptyset = \{w \in \omega \mid \text{patient has no symptoms in } w\}$
- $a_1 = \{w \in \omega \mid \text{patient has only symptom } S_1 \text{ in } w\}$
- $a_2 = \{w \in \omega \mid \text{patient has only symptom } S_2 \text{ in } w\}$
- $a_{12} = \{w \in \omega \mid \text{patient has both symptoms in } w\}$

That is, in this case the support-set of μ_1 is not generated by a single state. Rather, it is generated by four different states:

$$[\mu_1] = \{a_\emptyset, a_1, a_2, a_{12}\}^{\downarrow}$$

This reflects the fact that settling a question, such as μ_1, does not amount to establishing a *specific* piece of information, as in the case of a statement, but rather to establishing *one of several* alternative pieces of informations—which we may think of as the various complete answers to the question.

We may thus think of μ_1 as describing not a specific piece of information, as in the case of a statement, but rather a *type* of information. The elements of this type are the states $a_\emptyset, a_1, a_2, a_{12}$ regarded, respectively, as: the information that the patient has no symptom, the information that the patient has only symptom S_1; the information that the patient has only symptom S_2; and the information that the patient has both symptoms. To say that μ_1 is settled in a state is to say that some piece of information of this type is available.

We may generalize these observations as follows. If T is a generator for the support-set $[\varphi]$, we may regard φ as describing the type of information T. For, φ is settled in a state s iff some piece of information $a \in T$ is available in s:

$$s \models \varphi \iff s \subseteq a \text{ for some } a \in T$$

We may then take the following to be the fundamental property that distinguishes statements from questions.

Definition 1 (Specificity)
We say that a sentence φ is *specific* in case $[\varphi]$ admits a singleton generator. Otherwise, we say that φ is *generic*.

[5] In algebraic jargon, the meaning of a statement is always a *principal* down-set of the space of information states.

Statements are specific, that is, they may be regarded as describing a specific piece of information. Questions, on the other hand, are generic: the proposition that they express is not generated by any single information state; they must thus be regarded as describing a non-singleton type of information.

In general, there will of course be many generators T for a proposition $[\varphi]$. However, many propositions have a unique *minimal* generator.

Definition 2 (Alternatives)
The alternatives for a sentence φ are the maximal info states supporting φ.

$$\textsc{Alt}(\varphi) = \{s \mid s \models \varphi \text{ and there is no } t \supset s \text{ such that } t \models \varphi\}$$

Proposition 1

– *If* $[\varphi] = \textsc{Alt}(\varphi)^{\downarrow}$*, then* $\textsc{Alt}(\varphi)$ *is the unique minimal generator for* $[\varphi]$*.*
– *If* $[\varphi] \neq \textsc{Alt}(\varphi)^{\downarrow}$*, then* $[\varphi]$ *has no minimal generator.*

If $[\varphi] = \textsc{Alt}(\varphi)^{\downarrow}$, we will say that a sentence φ is *normal*. Statements are always normal, since we have $[\alpha] = \{|\alpha|\}^{\downarrow} = \textsc{Alt}(\alpha)^{\downarrow}$. The questions in our example are normal as well, and so are all the questions expressible in the propositional logic described in Section 3. The proposition ensures that each normal sentence φ can be construed in a canonical way as describing the type of information $\textsc{Alt}(\varphi)$.

Summing up, then, the support-conditional approach allows us to think of sentences as describing *information types*; a sentence is settled in a state iff some information of the corresponding type is available. Statements can be taken to describe singleton types, consisting of a unique piece of information; questions, on the other hand, always describe proper types, consisting of several distinct pieces of information.[6]

2.4 Logical entailment

In the truth-conditional approach, entailment is interpreted as preservation of truth: a conclusion follows from a set of premises if it is true whenever all the premisses are true. As a consequence, it is only statements, whose meaning can be captured in terms of truth-conditions, that can meaningfully figure in an entailment relation. In the support-conditional approach, entailment is interpreted as preservation of support: a conclusion follows from a set of premises if it is settled whenever all the premisses are. In symbols:

$$\Phi \models \psi \iff \text{for all } s \subseteq \omega: \ s \models \Phi \text{ implies } s \models \psi$$

where $s \models \Phi$ is shorthand for '$s \models \varphi$ for all $\varphi \in \Phi$'. As we saw, support-conditions are meaningful not only for statements, but also for questions. As

[6] This excludes tautological questions like *whether John is John*, which are trivially settled in all info states. We set these aside here as a manifestation of the familiar difficulties that the possible world framework has in dealing with tautologies and contradictions.

a consequence, sentences of both categories may now figure in an entailment relation. Thus, support semantics allows for a substantial generalization of the standard truth-conditional notion of entailment.

Now, given a sentence φ, let $T\varphi$ be an arbitrary generator for $[\varphi]$, so that we can think of φ as describing the type of information $T\varphi$.[7] Then, it is easy to see that the entailment $\varphi \models \psi$ holds iff any information of type $T\varphi$ yields some corresponding information of type $T\psi$.

$$\varphi \models \psi \iff \text{for every } a \in T\varphi \text{ there exists } a' \in T\psi \text{ such that } a \subseteq a'$$

In the case of multiple premisses, this generalizes as follows: $\varphi_1, \dots, \varphi_n \models \psi$ holds in case combining information of type $T\varphi_i$ for $1 \leq i \leq n$ is guaranteed to yield some information of type $T\psi$.

$$\begin{aligned}\varphi_1, \dots, \varphi_n \models \psi \iff & \text{ for every } a_1 \in T\varphi_1, \dots, a_n \in T\varphi_n \\ & \text{ there exists } a' \in T\psi \text{ such that } a_1 \cap \dots \cap a_n \subseteq a'\end{aligned}$$

To get acquainted with the significance of this generalized entailment relation, consider the case of a single premise. We have four possible entailment patterns: statement to statement, statement to question, question to statement, and question to question. Let us examine briefly the significance of each case.

- If α and β are statements, then $\alpha \models \beta$ expresses the fact that settling that α implies settling that β. This simply amounts to $|\alpha| \subseteq |\beta|$, that is, the information that α is true yields the information that β is true. As we pointed out above, this simply coincides with the usual truth-conditional notion of entailment.
- If α is a statement and μ a question, $\alpha \models \mu$ expresses the fact that settling that α implies settling μ. We may read $\alpha \models \mu$ as "α *logically resolves* μ". E.g., the statement *Galileo discovered Jupiter's moons* entails the question *whether Galileo discovered anything.*
  It is easy to see that we have $\alpha \models \mu \iff |\alpha| \subseteq a$ for some $a \in T\mu$: that is, α entails μ if the information that α yields some information of type μ.
- If μ is a question and α is a statement, then $\mu \models \alpha$ expresses the fact that whenever we settle μ—in any possible way—we also settle that α; in other words, it is impossible to resolve the question without establishing that α. We may read $\mu \models \alpha$ as "μ *presupposes* α". E.g., the question *in what year Galileo discovered Jupiter's moons* entails the statement *Galileo discovered Jupiter's moons.*
  It is easy to see that we have $\mu \models \alpha \iff a \subseteq |\alpha|$ for all $a \in T\mu$: that is, μ entails α iff any information of type μ yields the information that α.
- If μ and ν are questions, $\mu \models \nu$ express the fact that settling μ implies settling ν. This is precisely the relation of dependency that we pointed

[7] If φ is normal, as in our examples, the canonical choice will be $T\varphi = \textsc{Alt}(\varphi)$, but this assumption will not be needed here.

out in our initial examples, but now in its purely logical version, since *all* worlds, not just some contextually relevant ones, are taken into account. We may read $\mu \models \nu$ as "μ logically determines ν". E.g., the question *when and where Galileo discovered Jupiter's moons* entails the question *when Galileo discovered Jupiter's moons.*
In terms of information types, we have $\mu \models \nu \iff$ for all $a \in T\mu$ there is $a' \in T\nu$ such that $a \subseteq a'$: that is μ entails ν if any piece of information of type μ yields some corresponding piece of information of type ν.

Thus, support semantics gives rise to an interesting generalization of classical entailment, which captures not only the logical connections existing between pieces of information (the standard consequence relation), but also those existing between pieces of information and types of information (resolution, presupposition), and between one type of information and another (dependency).

2.5 Entailment in context

When we think about a statement being a consequence of another, it is rarely the purely *logical* notion of consequence that we are concerned with. Rather, we typically take some facts for granted, and then assess whether *on that basis*, the truth of one statement implies the truth of the other. We say, e.g., that *Galileo discovered some celestial body* is a consequence of *Galileo discovered Jupiter's moons*; in doing so, we take for granted that Jupiter's moons are celestial bodies: worlds in which this is not the case are not taken into account.

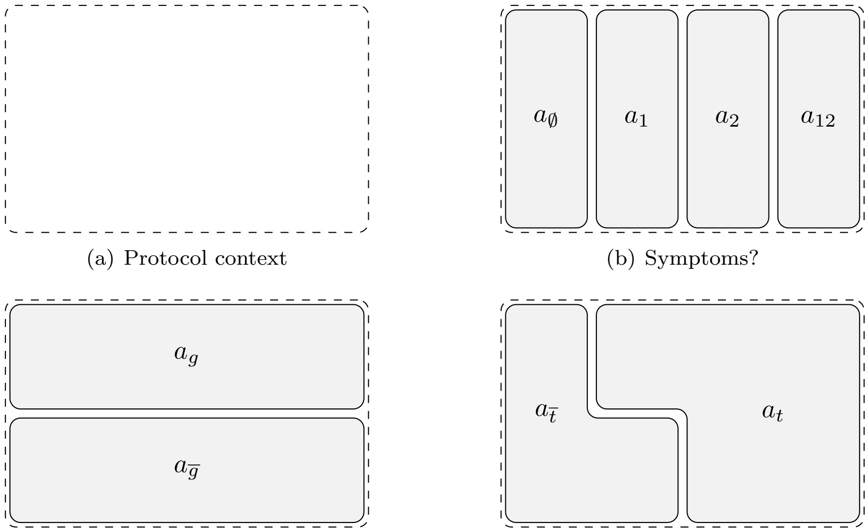
The same holds for questions: when we are concerned with dependencies, it is rarely purely *logical* dependencies that are at stake. Rather, we are usually concerned with the relations that one question bears to another, given certain background facts about the world. In our initial example, for instance, it is the hospital's protocol that provides the context relative to which the dependency holds. It is only within this specific context that the three questions in the example are linked by any interesting relations.

In order to capture these relations, besides the absolute notion of logical entailment that we discussed, we will also introduce a relativized notion of *contextual* entailment. We will model a context simply as an information state s. In assessing entailment relative to s, we take the information embodied by s for granted. This means that, to decide whether an entailment holds or not, only worlds in s, and states consisting of such worlds, are taken into account. Formally, we make the following definition.

$$\Phi \models_s \psi \iff \text{for all } t \subseteq s : \; t \models \Phi \text{ implies } t \models \psi$$

Contextual entailment captures relations of consequence, resolution, presupposition, and dependency which hold not purely logically, but against the background of a specific context.

Focusing on dependency, let us look in particular at how our initial example is captured as an instance of entailment in context. Let s denote our hospital



(a) Protocol context (b) Symptoms?

(c) Condition? (d) Treatment?

Fig. 1 The meanings of the three questions involved in our initial example, within the context s provided by the hospital's protocol. The sets displayed in the figures are the intersections of the alternatives for the questions with the context. To avoid clutter, we label each of these sets by the name of the corresponding alternative a. In fact, what is displayed is the intersection $a \cap s$ of this alternative with the context.

protocol context, which consists of the set of worlds which are compatible with the protocol. Thus, e.g., s contains worlds where the patient has both symptoms and the treatment is prescribed, but not worlds where the patient has both symptoms and the treatment is *not* prescribed, since such worlds are incompatible with the protocol.

Now, we saw that a state $t \subseteq s$ settles the question μ_1 of what symptoms the patient has in case it is included in one of the following four states, whose intersection with s is depicted in figure 1(b):

- $a_\emptyset = \{w \in \omega \mid \text{patient has no symptoms in } w\}$
- $a_1 = \{w \in \omega \mid \text{patient has only symptom } S_1 \text{ in } w\}$
- $a_2 = \{w \in \omega \mid \text{patient has only symptom } S_2 \text{ in } w\}$
- $a_{12} = \{w \in \omega \mid \text{patient has both symptoms in } w\}$

A state $t \subseteq s$ settles the question μ_2 of whether the patient is in good condition in case it settles that the patient *is* in good condition, or it settles that the patient is *not* in good condition. This holds just in case t is included in either of the following states, whose intersection with s is depicted in Figure 1(c):

- $a_g = \{w \in \omega \mid \text{patient is in good condition in } w\}$
- $a_{\overline{g}} = \{w \in \omega \mid \text{patient is not in good condition in } w\}$

Finally, a state $t \subseteq s$ settles the question ν of whether the treatment is prescribed just in case it settles that the treatment is prescribed, or it settles that the treatment is not prescribed. That is, in case t is included in one of the following two states, whose intersection with s is depicted in Figure 1(d):

- $a_t = \{w \in \omega \mid \text{treatment is prescribed in } w\}$
- $a_{\overline{t}} = \{w \in \omega \mid \text{treatment is not prescribed in } w\}$

Now, clearly, relative to the context s, neither μ_1 nor μ_2 by itself entails ν. For instance, μ_1 is settled in the state $a_1 \cap s$, but ν is not. This corresponds to the fact that the information that the patient has only symptom S_1 is not sufficient to determine whether the treatment is prescribed or not. Similarly, μ_2 is settled in each of the states $a_g \cap s$ and $a_{\overline{g}} \cap s$, but ν is not. This corresponds to the fact that information as to whether the patient is in good condition is not sufficient to determine whether the treatment is prescribed.

Hence, we have $\mu_1 \not\models_s \nu$ and $\mu_2 \not\models_s \nu$, which captures the fact that whether the treatment is prescribed is not fully determined by either the patient's symptoms or the patient's condition in the given context.

At the same time, μ_1 and μ_2 together *do* entail ν relative to s. For, consider a state $t \subseteq s$ which settles both μ_1 and μ_2: since t settles μ_1, t must be included in one of the sets $a_\emptyset$, a_1, a_2, a_{12}; and since t settles μ_2, t must be included in one of among a_g and $a_{\overline{g}}$. It is clear by inspecting the figure that any such state must be included in one among a_t and $a_{\overline{t}}$, which means that it also settles ν. Thus, we have $\mu_1, \mu_2 \models_s \nu$, which captures the fact that whether the treatment is prescribed is jointly determined by the patient's symptoms and condition. In this way, the dependence relation of our initial example is captured as a particular instance of entailment—more precisely, as a case of question entailment in context.

It is also natural to look at this relation in terms of information types. Since all three questions involved in the examples are normal, we can associate them to the following information types.

- $\textsc{Alt}(\mu_1) = \{a_\emptyset, a_1, a_2, a_{12}\}$
- $\textsc{Alt}(\mu_2) = \{a_g, a_{\overline{g}}\}$
- $\textsc{Alt}(\nu) = \{a_t, a_{\overline{t}}\}$

Then, the entailment $\mu_1, \mu_2 \models_s \nu$ amounts to the following.

$$\mu_1, \mu_2 \models_s \nu \iff \begin{array}{l} \text{for any } a \in \textsc{Alt}(\mu_1) \text{ and any } a' \in \textsc{Alt}(\mu_2) \\ \text{there is } a'' \in \textsc{Alt}(\nu) \text{ such that } s \cap a \cap a' \subseteq a'' \end{array}$$

That is, the entailment holds if, within the context s provided by the protocol, combining a piece of information of type *symptoms* ($a \in \textsc{Alt}(\mu_1)$) with one of type *conditions* ($a' \in \textsc{Alt}(\mu_2)$) is bound to yield some piece of information of type *treatment* ($a'' \in \textsc{Alt}(\nu)$). This shows how the contextual entailment $\mu_1, \mu_2 \models_s \nu$ captures precisely the relation that we observed to exist between the types of information $\textsc{Alt}(\mu_1)$, $\textsc{Alt}(\mu_2)$, and $\textsc{Alt}(\nu)$ within the context s.

2.6 From contextual to logical entailment

Contextual entailments can be made into *logical* entailments by turning the relevant contextual material into an explicit premise. If Γ is a set of statements, and if $|\Gamma|$ is the set of worlds at which these statements are all true, we have the following connection:

$$\Phi \models_{|\Gamma|} \psi \iff \Gamma, \Phi \models \psi$$

That is, if a context s is describable by a set Γ of statements, contextual entailment relative to s amounts to logical entailment with the statements in Γ as additional premises. In our example, the context s may be described by means of a statement such as the following.

γ. The treatment is prescribed if and only if the patient has symptom S_2, or the patient has symptom S_1 and is in good physical condition.

Thus, the dependency in our example is not only captured by the contextual entailment $\mu_1, \mu_2 \models_s \nu$ relative to the protocol context, but also by its purely logical counterpart $\gamma, \mu_1, \mu_2 \models \nu$ in which the hospital's protocol is turned into an additional premise.

2.7 Internalizing entailment

In support semantics, the contexts to which entailment can be relativized are the same kind of objects at which sentences are evaluated, namely, information states. This ensures that a support-based logic can always be enriched with an operation of *implication* which internalizes the meta-language relation of entailment. In other words, a logical system whose semantics is given in terms of support may always be equipped with a connective $\to$ such that, for any sentences φ and ψ, $\varphi \to \psi$ is settled in s iff φ entails ψ relative to s.

$$s \models \varphi \to \psi \iff \varphi \models_s \psi$$

Simply by making explicit what the condition $\varphi \models_s \psi$ amounts to, we get the inductive support clause governing this operation:

$$s \models \varphi \to \psi \iff \text{for all } t \subseteq s: \; t \models \varphi \text{ implies } t \models \psi$$

Interestingly, this is, *mutatis mutandis*, precisely the interpretation of implication that we find in virtually all information-based semantics.

If we apply this clause to statements, what we get is simply the usual material conditional of classical logic. For, using Relation 1 we have:

$$\begin{aligned} s \models \alpha \to \beta &\iff \forall t \subseteq s,\; t \models \alpha \text{ implies } t \models \beta \\ &\iff \forall t \subseteq s,\; t \subseteq |\alpha| \text{ implies } t \subseteq |\beta| \\ &\iff s \cap |\alpha| \subseteq |\beta| \\ &\iff s \subseteq \overline{|\alpha|} \cup |\beta| \end{aligned}$$

where $\overline{|\alpha|} = \omega - |\alpha|$ is the set of worlds where α is false. Thus, the conditional $\alpha \to \beta$ is supported in a state s iff the corresponding material conditional is true everywhere in s. This is interesting, as it shows that the standard material conditional may be seen as arising precisely by internalizing within the language the relation of contextual entailment between statements.

What is more interesting, however, is that the clause above defines an operation which *generalizes* the material conditional. For, we saw that support semantics is suitable for interpreting questions, besides statements. If our language contains questions, implication among them is naturally defined: given two questions μ and ν, we thus have a formula $\mu \to \nu$ which is supported by a state s in case μ determines ν relative to s.

What this shows is that the support approach does not only allow us to generalize the relation of entailment to questions, capturing dependencies: it also allows us to generalize in a parallel way the conditional operator to questions, enabling us to express these dependencies within the language.

2.8 Summing up

We have seen that classical logic can be given an alternative, informational semantics in terms of *support conditions*, which determines when a sentence is *settled* by a body of information, rather than when it is true at a world. Unlike truth-conditional semantics, support semantics can interpret questions in a natural way. In this approach, a formula may be regarded as describing a type of information: statements describe singleton types, which may be identified with specific pieces of information; questions describe non-singleton types, which are instantiated by several different pieces of information.

This unified semantic account of statements and questions allows for a generalization of the truth-conditional notion of entailment: while entailments among statements have the usual significance, entailments involving questions capture dependencies. In particular, an entailment of the form $\alpha, \mu \models \nu$ captures the fact that, in the context described by the statement α, the question μ determines the question ν: that is, given the information that α, any information of type μ yields some corresponding information of type ν.

3 Questions in propositional logic

In this section, the ideas discussed abstractly so far will be illustrated by means of a concrete logical system. We will do this in the simplest possible setting, that of propositional logic. The system that we will discuss is essentially the system InqB of propositional inquisitive logic (Ciardelli, 2009; Groenendijk and Roelofsen, 2009; Ciardelli and Roelofsen, 2011). However, we will take a new perspective on this system. In previous work, the idea was that standard propositional formulas are given a more fine-grained semantics, adding an inquisitive dimension to purely truth-conditional meaning: as a consequence, InqB emerged as a non-classical logic. Here, we will instead take the perspective described in the previous section: we will first re-implement classical logic in a support-based style, and then extend this classical core by a question-forming disjunction connective: as a consequence, InqB will now emerge as a *conservative extension* of classical propositional logic. Technically, the difference between the two perspectives is merely notational. Conceptually, however, the difference is an important one, and we will see that many features of inquisitive logic take on a clearer intuitive significance from the present perspective. Besides, the mere possibility of this alternative perspective is an interesting feature of InqB, which is worth exploring in some detail.

For proofs of the technical results mentioned in this section, the reader is referred to Ciardelli (2009) and Ciardelli and Roelofsen (2011). Occasionally, proofs are provided for claims which lack a direct analogue in the literature.

3.1 Propositional information states

Let $\mathcal{P}$ be a set of propositional atoms. We will take a possible world to be a propositional valuation, that is, a function $w : \mathcal{P} \to \{0, 1\}$ which specifies which atoms are true and which are false. As a consequence, *information states* will be sets of propositional valuations. The set ω containing *all* valuations represents the trivial state in which no information is present. At the opposite end of the spectrum, the empty set $\emptyset$ represents the state of inconsistent information; non-empty states will be referred to as *consistent states.*

3.2 Support semantics for classical propositional logic

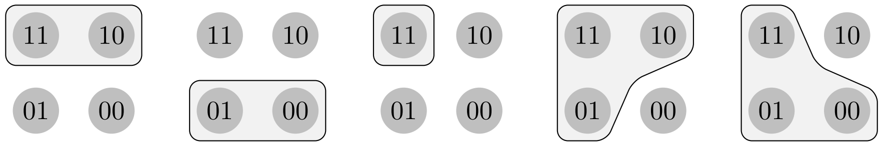
Let us start by providing a support semantics for classical propositional logic. The set $\mathcal{L}_c$ of *classical formulas* will consist of propositional formulas built up from atoms and the *falsum* constant $\bot$ by means of the primitive connectives $\wedge$ and $\to$. We will take negation and classical disjunction to be defined from these primitive connectives as follows:

$$\neg\varphi \;:=\; \varphi \to \bot \qquad \varphi \vee \psi \;:=\; \neg(\neg\varphi \wedge \neg\psi)$$

The relation of support between information states $s \subseteq \omega$ and classical formulas $\varphi \in \mathcal{L}_c$ is defined by the following clauses.

Definition 3 (Support)

- $s \models p \iff w(p) = 1$ for all $w \in s$
- $s \models \bot \iff s = \emptyset$
- $s \models \varphi \wedge \varphi \iff s \models \varphi$ and $s \models \psi$
- $s \models \varphi \to \psi \iff$ for all $t \subseteq s$, $t \models \varphi$ implies $t \models \psi$



(a) p (b) $\neg p$ (c) $p \wedge q$ (d) $p \vee q$ (e) $p \rightarrow q$

Fig. 2 The alternatives for some classical formulas. 11 represents a words where p and q are both true, 10 a world where p is true and q is false, etc. As ensured by Proposition 3, each formula has a unique alternative, which coincides with its truth-set.

The clauses may be read as follows. An atom p is settled in s in case it is true at every world in s. The falsum constant $\bot$ is only settled in the inconsistent state, $\emptyset$. A conjunction is settled in s in case both conjuncts are. Finally, implication internalizes entailment in the way described in the previous section: an implication is settled in s in case the antecedent entails the consequen relative to s; that is, in case enhancing s so as to settle the antecedent is guaranteed to lead to a state which also settles the consequent.

We will say that a state s is *compatible* with a formula φ, notation $s \between \varphi$, in case s can be enhanced consistently to a state that supports φ:

$$s \between \varphi \iff t \models \varphi \text{ for some consistent } t \subseteq s$$

Using this notion, the derived semantic clauses for negation and disjunction may be expressed as follows.

– $s \models \neg\varphi \iff$ it is not the case that $s \between \varphi$
– $s \models \varphi \vee \psi \iff$ for all consistent $t \subseteq s$, $t \between \varphi$ or $t \between \psi$

That is, a negation $\neg\varphi$ is settled in s in case s is *in*compatible with φ, i.e., in case s cannot be consistently enhanced to support φ. As for disjunction, $\varphi \vee \psi$ is settled in s if any consistent enhancement of s is bound to be compatible with either φ or ψ—that is, if s cannot be consistently enhanced to a state which is incompatible with both disjuncts.

Now we can verify inductively that, indeed, support-conditions are related to truth-conditions according to Relation 1 above: a classical formula φ is supported by a state s if and only if it is true at every world in s.

Proposition 2 *For any state $s \subseteq \omega$ and any $\varphi \in \mathcal{L}_c$: $s \models \varphi \iff s \subseteq |\varphi|$, where $|\varphi|$ is the set of worlds at which φ is true, as given by classical logic.*

Notice that this property can be rewritten as follows: $[\varphi] = \{|\varphi|\}^{\downarrow}$. Thus, any classical formula is *specific* in the sense of Definition 1, and may thus be regarded as a statement. An immediate consequence of this Proposition is that a classical formula always has a unique alternative, which coincides with its truth-set. This is illustrated by Figure 2.

Proposition 3 *For any classical formula φ, $\textsc{Alt}(\varphi) = \{|\varphi|\}$.*

The following three properties of the semantics also follow immediately from Proposition 2.

Proposition 4 *For any classical formula φ, we have:*

Persistence property: if $s \models \varphi$ and $t \subseteq s$, then $t \models \varphi$
Empty state property: $\emptyset \models \varphi$
Regularity property: if $s \models \varphi$ for every $s \in S$, then $\bigcup S \models \varphi$

Proposition 2 shows that, for formulas $\varphi \in \mathcal{L}_c$, the support-semantics we just described is equivalent to the standard truth-conditional semantics. Thus, what we have provided so far is simply an alternative semantic foundation for classical propositional logic. However, as we discussed above, there is an important advantage to the support implementation: it opens up the possibility to interpret formulas that represent *questions*, rather than statements.

3.3 Adding questions to propositional logic

Let us extend our classical language with a new connective $\vee\!\!\!\vee$, called *inquisitive disjunction*. Intuitively, $\varphi \vee\!\!\!\vee \psi$ will stand for the question *whether φ or ψ*, which is settled just in case one among φ and ψ is settled.[8]

Definition 4 (Support for inquisitive disjunction)

– $s \models \varphi \vee\!\!\!\vee \psi \iff s \models \varphi$ or $s \models \psi$

If $\varphi \in \mathcal{L}_c$, the polar question *whether φ*, which is settled in case either φ or $\neg\varphi$ is settled, can be expressed by means of inquisitive disjunction as $\varphi \vee\!\!\!\vee \neg\varphi$. Thus, while the classical disjunction $p \vee \neg p$ is a tautology, supported by any state, the inquisitive disjunction $p \vee\!\!\!\vee \neg p$ expresses the polar question *whether p*. For convenience, we will make use the following abbreviation:

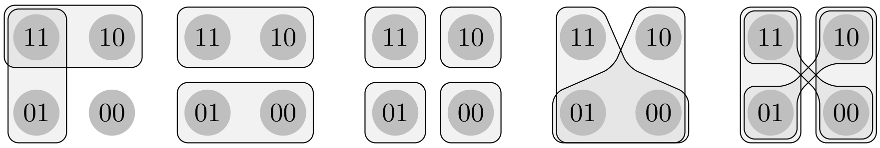
– $?\varphi := \varphi \vee\!\!\!\vee \neg\varphi$

The full language $\mathcal{L}$ of our system InqB is the propositional language generated from propositional atoms and $\bot$ by means of the connectives $\wedge, \rightarrow$, and $\vee\!\!\!\vee$.

$$\varphi \quad ::= \quad p \,|\, \bot \,|\, \varphi \wedge \varphi \,|\, \varphi \rightarrow \varphi \,|\, \varphi \vee\!\!\!\vee \varphi$$

For this extended language, the *persistence property* of Proposition 4 still holds, as one can easily verify by induction: if a formula is settled in a state, it remains settled in any extension of the state. The *empty state property* holds as well: in the inconsistent information state, every formula is settled. However,

[8] It is quite possible that, in natural language, the alternative question *whether φ or ψ* is only settled if we establish which one of φ and ψ holds, *to the exclusion of the other*. If so, then such a question should be translated in our formal language not as $\varphi \vee\!\!\!\vee \psi$, but as an *exclusive* inquisitive disjunction $\varphi \underline{\vee\!\!\!\vee} \psi := (\varphi \wedge \neg\psi) \vee\!\!\!\vee (\psi \wedge \neg\varphi)$. Nothing important in this paper hinges on this empirical issue. What matters is that, however construed, such questions can be represented and reasoned about in the system.



(a) $p \vee q$ (b) $?p$ (c) $?p \wedge ?q$ (d) $p \to ?q$ (e) $?p \to ?q$

Fig. 3 The alternatives for some questions in InqB.

the *regularity* property fails, in general, for formulas containing inquisitive disjunction; that is, support is no longer preserved under unions. To see this, consider the polar question $?p := p \vee \neg p$: clearly, $?p$ is settled at the state $|p|$ and also at the state $|\neg p|$, but not at their union $|p| \cup |\neg p| = \omega$, which amounts to the set of all possible worlds. Moreover, as we expect from our discussion in the previous section, formulas containing inquisitive disjunction are not in general specific, that is, they do not always admit a singleton generator. For instance, it is easy to see that the proposition $[?p]$, whose maximal elements are depicted in Figure 3(b), is not generated by any singleton. As suggested by our discussion in the previous section, we will take specificity and genericity to be the defining features of statements and questions, respectively.[9,10]

Definition 5 (Statements and questions)

- A formula φ is a statement iff it is specific.
- A formula φ is a question iff it is generic.

[9] This is not the usual inquisitive semantics terminology. In most previous work (e.g., Ciardelli et al., 2012) specific sentences are called *assertions*; generic sentences are called *inquisitive*, while the term *question* is reserved for formulas whose support-set covers the whole logical space. We use *statement* instead of *assertion* here, since the latter term is often used to refer to a speech act. As for the term *question*, the reason we go for a more liberal notion is that we want to consider questions that can only be resolved in *some* worlds. Indeed, here we take sentences with multiple alternatives which do not cover the whole logical space to correspond to questions like (i-a), which can only truthfully resolved in worlds in which Mary is coming. In previous work, the same sentences were taken to correspond to hybrids such as (i-b).

(i) a. Is Mary coming with or without John?
b. Mary is coming, but is John coming as well?

From the present perspective, hybrids like (i-b) cannot be represented in the system, while from the perspective of previous work, it is questions like (i-a) that cannot be represented. In the end, the idea is that the both kinds of sentences can be represented in a system which explicitly captures presuppositions as a meaning component, as described for instance in §7.1 of Ciardelli et al. (2012). Then, while (i-a) and (i-b) have the same support conditions, they can still be distinguished in terms of their presuppositions: (i-a) presupposes that Mary is coming, while (i-b) presupposes nothing.

[10] Notice that this definition partitions the formulas of our language into statements and questions. In this respect, our perspective brings InqB closer to the *dichotomous inquisitive semantics* InqD_π of Ciardelli et al. (2015). The difference is that, here, the dichotomy is not built into the syntax of the language, and no restrictions are placed on the applicability of propositional connectives, which makes the logic more natural.

Henceforth, the meta-variables α, β range over statements, μ, ν range over questions, and φ, ψ, χ range over arbitrary formulas.

Formulas in $\mathcal{L}$ are always normal, that is, the set of alternative for a formula are always a generator for the formula's support-set.

Proposition 5 (Normality) *For all* $\varphi \in \mathcal{L}$, $[\varphi] = \textsc{Alt}(\varphi)^{\downarrow}$

This means that we may regard a formula φ in a canonical way as denoting the type of information $\textsc{Alt}(\varphi)$. This allows us to give a very visual alternative characterization of the classes of statements and questions.

Proposition 6

- *A formula* φ *is a statement iff it has a unique alternative.*
- *A formula* φ *is a question iff it has two or more alternatives.*

Let us first focus on the class of statements. An interesting observation is that statements may be characterized as formulas whose semantics is completely determined at the level of singleton states.

Proposition 7 φ *is a statement iff the following holds for all* s*:*

$$s \models \varphi \iff \{w\} \models \varphi \text{ for all } w \in s$$

Notice that we may generalize the notion of truth from classical formulas to the whole language $\mathcal{L}$ by defining it in terms of support at singleton states: φ is true at w if it is supported at $\{w\}$. Then, the previous proposition says that statements are precisely the truth-conditional formulas in the language.

Next, recall that Proposition 2 guarantees that any classical formula is a statement. Conversely, any statement is equivalent to a classical formula, which shows that, by adding $\vee\!\!\!\vee$, we are enabling our logic to express questions, but not to express new statements.[11] To prove this, it will be useful to associate to any formula a corresponding classical formula.

Definition 6 (Classical variant of a formula)
The classical variant φ^{cl} of a formula φ is obtained from φ by replacing all occurrences of inquisitive disjunction by classical disjunction.

Now, for any formula φ, its classical variant φ^{cl} is a classical formula having as its unique alternative the union of all the alternatives for φ.

Proposition 8 *For any* φ, $\textsc{Alt}(\varphi^{cl}) = \{\bigcup \textsc{Alt}(\varphi)\}$.

This proposition implies that a formula φ is a statement iff it is equivalent to its classical variant φ^{cl}. As a consequence, we get the following corollary.

[11] This is not necessarily the case for richer languages. In the *inquisitive epistemic logic* of Ciardelli and Roelofsen (2015), questions may be embedded under modalities, resulting in new statements expressing, for instance, that an agent *wonders whether p*. In such a system, the presence of questions also enables the language to express statements that have no equivalent counterpart in a classical language.

Corollary 1 *Any statement is equivalent to a classical formula.*

Another important observation is that, by the semantic clause for negation, $\neg\varphi$ is always a statement. In particular, the double negation of a formula is always a statement, which is equivalent to the classical variant of the formula.

Proposition 9 $\neg\neg\varphi \equiv \varphi^{cl}$

As a consequence, statements can also be characterized as being precisely those formulas which are equivalent to their own double negation.

Proposition 10 $\varphi \equiv \neg\neg\varphi \iff \varphi$ *is a statement.*

Let us now turn our attention to questions. An important asset of InqB is that the truth-conditional operations of conjunction and implication are generalized so that they can operate not only on statements, but also on questions. Consider first conjunction: while conjunction gives the standard results when applied to statements (cf. Figure 2(c)) it can now be applied also to two questions, such as $?p$ and $?q$. This results in a question $?p \wedge ?q$ which is settled if and only if both conjuncts are settled, as illustrated in Figure 3(c).

Now consider implication: we saw that an implication $\varphi \to \psi$ is supported in a state s iff the entailment $\varphi \models_s \psi$ holds. When applied to two statements, implication gives the standard results, as discussed in Section 2.7. However, implication can now also be applied to questions. As an example, consider the formula $p \to ?q$. As illustrated by Figure 3(d), this formula has two alternatives, corresponding to $p \to q$ and $p \to \neg q$. Thus, the implication $p \to ?q$ renders the meaning of natural language questions like (2).[12]

(2) If John invites Mary to the party, will she go?

Next, consider an implication having questions both as consequent and as antecedent, such as $?p \to ?q$: this formula is supported in s if $?p \models_s ?q$, that is, in case $?p$ determines $?q$ relative to s. Thus, $?p \to ?q$ is a question that asks for enough information to establish a dependency of $?q$ on $?p$. As illustrated by Figure 3(e), this question has four alternatives, corresponding to the four ways in which such a dependency may obtain:[13]

1. $(p \to q) \wedge (\neg p \to q) \equiv q$
2. $(p \to q) \wedge (\neg p \to \neg q) \equiv q \leftrightarrow p$
3. $(p \to \neg q) \wedge (\neg p \to q) \equiv q \leftrightarrow \neg p$
4. $(p \to \neg q) \wedge (\neg p \to \neg q) \equiv \neg q$

[12] The fact that the effect of conjunction and implication on questions does not have to be stipulated, but follows from the same clauses that govern these connectives in statements is a remarkable feature of InqB. In previous logics of questions, like Belnap and Steel (1976), analogous outcomes were obtained from operations defined ad-hoc, which bore no systematic relation to their truth-conditional counterpart.

[13] Such conditional questions are not directly expressible as conditionals in English. A reasonable rendering is a sentence like: *given a patient's symptoms and conditions, when do we administer the treatment?* Notice that the existence of the dependency we pointed out in our example amounts precisely to the fact that the hospital protocol settles this question.

3.4 Resolutions and inquisitive normal form

An important logical feature of the system InqB is that we can compute, recursively on the structure of a formula φ, a set of classical formulas which provide can be taken to name the different pieces of information of type φ. We refer to these formulas as the *resolutions* of φ.

Definition 7 (Resolutions)

- $\mathcal{R}(p) = \{p\}$
- $\mathcal{R}(\bot) = \{\bot\}$
- $\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\varphi) \text{ and } \beta \in \mathcal{R}(\psi)\}$
- $\mathcal{R}(\varphi \to \psi) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)} (\alpha \to f(\alpha)) \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$
- $\mathcal{R}(\varphi \vee\!\!\!\vee \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$

Clearly, resolutions are by definition classical formulas. Moreover, it is easy to show by induction that any classical formula is the only resolution of itself.

Proposition 11 *If $\alpha \in \mathcal{L}_c$, then $\mathcal{R}(\alpha) = \{\alpha\}$.*

The essential property of resolutions is given by the following Proposition: to settle the formula φ is to establish that α is true, for some resolution α of φ.

Proposition 12 *For any formula $\varphi \in \mathcal{L}$ and any state $s \subseteq \omega$:*

$$s \models \varphi \iff s \models \alpha \textit{ for some } \alpha \in \mathcal{R}(\varphi)$$

Now, let us write $|\mathcal{R}(\varphi)|$ for the set of truth-sets of resolutions of φ, that is, $|\mathcal{R}(\varphi)| = \{|\alpha| \mid \alpha \in \mathcal{R}(\varphi)\}$. The above Proposition can be restated as follows:

$$[\varphi] = |\mathcal{R}(\varphi)|^{\downarrow}$$

This means that a formula $\varphi \in \mathcal{L}$ can always be thought of as capturing a type of information whose elements are named by the resolutions of φ.[14] As a corollary of this fact, we have the following normal form result.

Proposition 13 (Inquisitive normal form)
Let $\varphi \in \mathcal{L}$ and let $\mathcal{R}(\varphi) = \{\alpha_1, \ldots, \alpha_n\}$. Then, $\varphi \equiv \alpha_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \alpha_n$

[14] Notice that, in general, resolutions do not correspond exactly to the alternatives for the formula. This is because one resolution may be strictly stronger than another, and thus it may fail to give rise to an alternative: for instance, $\bot$ is a resolution of $p \vee\!\!\!\vee \bot$, but $|\bot| = \emptyset$ is not an alternative for $p \vee\!\!\!\vee \bot$. If we wanted to obtain a perfect correspondence between resolutions and alternatives, we could filter out from $\mathcal{R}(\varphi)$ those resolutions that strictly entail another resolution. This is unproblematic, but it is also not needed for our purposes.

3.5 Inquisitive entailment

Now that we have set up a logical system encompassing both statements and questions, let us take a look at the relation of entailment which arises from it. As we expect, on the classical fragment of the language, this relation coincides with truth-conditional entailment in classical propositional logic, denoted CPL.

Proposition 14 (Entailment among classical formulas is classical) *Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}_c$. Then $\Gamma \models \alpha \iff \Gamma$ entails α in CPL.*

In this precise sense, InqB is a conservative extension of CPL from our perspective. This is interesting because, simply by taking $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ to be the "official" disjunction of the system, rather than a new connective, InqB may also be regarded as a special kind of intermediate logic (as it has been, in Ciardelli, 2009; Ciardelli and Roelofsen, 2011).

Let us now consider cases of entailments involving questions. First, recall that an entailment $\alpha \models \mu$ from a statement to a question holds if α *logically resolves* μ. As an illustration, we have $p \wedge q \models ?p$, but $p \vee q \not\models ?p$: the question $?p$ is resolved by the information that $p \wedge q$, but not by the information that $p \vee q$. Indeed, the following proposition says that a statement entails a question iff it entails some specific resolution of it. This holds not only for logical entailment, but also for entailment relative to an arbitrary context. The purely logical case may be obtained by setting $s = \omega$.

Proposition 15 $\alpha \models_s \mu \iff \alpha \models_s \beta$ *for some* $\beta \in \mathcal{R}(\mu)$*.*

Next, consider an entailment $\mu \models \alpha$ from a question to a statement. We said that such an entailment captures the fact that μ *logically presupposes* α, i.e., that μ can only be resolved provided α is true. As an illustration, we have $p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} q \models p \vee q$, but $?p \not\models p \vee q$: the information that $p \vee q$ is presupposed by the question $p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} q$, but not by the question $?p$. Indeed, the following proposition shows that α is entailed by μ iff it is entailed by the disjunction $\bigvee \mathcal{R}(\mu)$ of the resolutions to μ, which may be seen as capturing the question's presupposition.

Proposition 16 $\mu \models_s \alpha \iff \bigvee \mathcal{R}(\mu) \models_s \alpha$

Finally, consider the most interesting case from our perspective, namely, entailment between questions. We saw in Section 2.4 that $\mu \models \nu$ captures the fact that ν is logically determined by μ. Moreover, we saw in Section 2.5 that adding a statement α as assumption, $\alpha, \mu \models \nu$ captures the fact that μ determines ν *in the context described by* α:

$$\alpha, \mu \models \nu \iff \mu \models_{|\alpha|} \nu$$

Of course, things are similar when we have several statements and questions as premises. As an illustration, let us look at how our initial example can be captured in InqB. We will make use of four propositional atoms: s_1 and s_2 stand for the statement that the patient presents the corresponding symptom; g stands for the statement that the patient is in good physical condition; finally,

t stands for the statement that the treatment is prescribed. The protocol of our hospital is encoded by the following classical formula:

$$\gamma \ := \ t \ \leftrightarrow \ s_2 \vee (s_1 \wedge g)$$

The question μ_1 of what symptoms (out of S_1 and S_2) the patient presents is captured by the formula $?s_1 \wedge ?s_2$. The question μ_2 of whether the patient is in good physical condition is captured by the formula $?g$, and the question ν of whether the treatment is prescribed is captured by the formula $?t$. Thus, the dependency which we observed to hold in our initial example is captured by the following entailment, which is seen valid in InqB.

$$\gamma, \ \ ?s_1 \wedge ?s_2, \ \ ?g \ \ \models \ \ ?t$$

3.6 Basic logical features of InqB

Now that we have discussed the significance of entailment in InqB, let us review some basic features of this logic. First, it is easy to see that Proposition 15 implies the following property, which plays an important role in axiomatizing the logic: if a statement entails an inquisitive disjunction, it must entail a specific disjunct. This is true not only for logical entailment, but also for entailment relative to an arbitrary context.

Proposition 17 (Split property)
Let α be a statement and s a state. If $\alpha \models_s \varphi \mathbin{\vee\!\!\!\vee} \psi$, then $\alpha \models_s \varphi$ or $\alpha \models_s \psi$.

As a corollary, InqB has the disjunction property for inquisitive disjunction: if $\varphi \mathbin{\vee\!\!\!\vee} \psi$ is logically valid, at least one of the disjuncts must be logically valid.

A second, important feature of InqB is the semantic deduction theorem, connected to the fact that implication internalizes entailment in context.

Proposition 18 (Semantic deduction theorem)
For any set of formulas Φ, and formulas ψ and χ: $\Phi, \psi \models \chi \iff \Phi \models \psi \to \chi$

Finally, recall that we have seen in Proposition 10 that a formula is a statement iff it is equivalent to its double negation. Since a formula always entails its own double negation, the previous proposition gives us the following fact.

Proposition 19 ($\neg\neg$-elimination characterizes statements)
For any $\varphi \in \mathcal{L}$: $\neg\neg\varphi \models \varphi \iff \varphi$ is a statement.

Indeed, since the semantics of statements is truth-conditional (Proposition 7), statements obey not only this law, but *all* the laws of classical logic.

Proposition 20 (Statements obey classical logic)
Let α be a statement, and let γ be a valid classical formula. Then $\models \gamma[\alpha/p]$.

Proof Since α is a statement, we have $\alpha \equiv \alpha^{cl}$, and thus also $\gamma[\alpha/p] \equiv \gamma[\alpha^{cl}/p]$. Since since γ and $\gamma[\alpha^{cl}/p]$ are classical formulas, and since classical logic is closed under substitution, from the assumption $\models \gamma$ it follows $\models \gamma[\alpha^{cl}/p]$. Finally, since $\gamma[\alpha/p] \equiv \gamma[\alpha^{cl}/p]$ we have $\models \gamma[\alpha/p]$. □

On the other hand, Proposition 19 shows that questions do *not*, in general, obey the laws of classical logic. However, the following fact—which follows from results in Ciardelli and Roelofsen (2011)—shows that all formulas, statements and questions alike, obey the weaker laws of *intuitionistic* logic.

Proposition 21 (All formulas obey intuitionistic logic)
Let $\varphi \in \mathcal{L}$, and let $\gamma \in \mathcal{L}_c$ be valid in intuitionistic logic. Then, $\models \gamma[\varphi/p]$.

Thus, the logic of questions has a constructive flavor. This also corresponds to something that we will see in a moment about proofs in InqB: while proofs involving only statements are simply proofs in classical logic, proofs involving questions have an interesting constructive interpretation.

4 Reasoning with questions

In the logic literature, the possibility and even the meaningfulness of reasoning with questions has been doubted, or even overtly denied. In the introduction to their book *The Logic of Questions and Answers*, Belnap and Steel (1976) write:

> Absolutely the wrong thing is to think [the logic of questions] is a logic in the sense of a deductive system, since one would then be driven to the pointless task of inventing an inferential scheme in which questions, or interrogatives, could serve as premises and conclusions.

This section is devoted to showing that Belnap and Steel were wrong: questions *do* have a role to play in deductive systems, and an interesting one at that. For, we have already seen that questions can meaningfully take part in entailment relations, and that entailments between questions capture interesting logical relations. Clearly, this calls for a deductive system involving questions. In this section, we show that it is possible to provide such a system, and that, moreover, questions have inferential features that are logically natural.

One may wonder, however, what sense it makes, conceptually, to make inferential moves such as assuming a question in a proof, inferring something from a question, or inferring a question from something else. The perspective described in this paper provides a natural answer to this question.

When in a proof we assume a statement α, we are supposing to have the information that α. When we assume a question μ, on the other hand, what we are supposing to have is *some* information of type μ. E.g., by assuming the question $?s_1 \wedge ?s_2$ of what symptoms the patient has, we are supposing to have a specification of the patient's symptoms, without however making any specific assumption as to what those symptoms are. When we make inferences from a question μ, we have to ensure that the conclusion follows no matter what specific information of type μ we are actually given: all we can rely on is the *type* of certain information, not the information itself. Finally, when we infer a question μ, this means is that, under the given assumptions, we have some information of type μ—though precisely *what* information we have may depend on the specific information that we are given for the assumptions.

We may thus regard a question μ in a proof as a *placeholder* for an arbitrary piece of information of the corresponding type. In this way, by using questions we can manipulate information that is *generic*, i.e., not completely specified. This may be useful for a number of reasons. In our example, e.g., we are not just concerned with a specific patient, whose symptoms and conditions are given once and for all; rather, we are interested in capturing a connection that holds generally, regardless of what the data relative to a specific patient are.

In this section we will illustrate these insights by providing a concrete proof system for InqB, in which both questions and statements may be manipulated.

4.1 A proof system for InqB

This sub-section describes a sound and complete proof system for InqB. Unlike the systems provided by Ciardelli and Roelofsen (2011), which are Hilbert-style, here we will work with a natural-deduction system. This allows for more insightful formal proofs, providing a better grasp of the significance of reasoning with questions. Since it is straightforward to adapt the techniques in Ciardelli and Roelofsen (2011) to this system, we will not bother with the issue of proving completeness. Instead, we will focus on an aspect which is not discussed in previous work: the conceptual significance of the inference rules, and of inquisitive proofs as a whole.

The rules of our natural deduction system are described in Figure 4. As customary, we refer to the introduction rule for a connective $\circ$ as $(\circ\mathsf{i})$, and to the elimination rule as $(\circ\mathsf{e})$. We write $P : \Phi \vdash \psi$ to mean that P is a proof whose set of undischarged assumptions is included in Φ, and whose conclusion is ψ, and we write $\Phi \vdash \psi$ to mean that a proof $P : \Phi \vdash \psi$ exists.

Let us comment briefly on each of the rules. First, notice that conjunction, implication, and the falsum constant are handled by their standard inference rules. This is true even when these connectives apply to questions. This shows that we can reason with conjunctions and implications among questions just as we normally reason with conjunction and implication. Also, notice that, since negation $\neg\varphi$ is defined as $\varphi \to \bot$, the usual rules for negation, described in Figure 5, are particular cases of the rules for implication.

Next, consider inquisitive disjunction: perhaps surprisingly, inquisitive disjunction is governed by the standard inference rules for disjunction. The introduction rules say that, if we settle a specific disjunct, then we have settled the inquisitive disjunction. Conversely, the elimination rule says that, if we can reach a conclusion from the assumption that we have settled φ and also from the assumption that we have settled ψ, then we can reach the same conclu-

Conjunction

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi \wedge \psi}{\varphi} \qquad \frac{\varphi \wedge \psi}{\psi}$$

Implication

$$\frac{\begin{array}{c}[\varphi] \\ \vdots \\ \psi\end{array}}{\varphi \to \psi} \qquad \frac{\varphi \quad \varphi \to \psi}{\psi}$$

Inquisitive disjunction

$$\frac{\varphi}{\varphi \mathbin{\vee\mkern-10mu\vee} \psi} \qquad \frac{\psi}{\varphi \mathbin{\vee\mkern-10mu\vee} \psi} \qquad \frac{\varphi \mathbin{\vee\mkern-10mu\vee} \psi \quad \begin{array}{c}[\varphi] \\ \vdots \\ \chi\end{array} \quad \begin{array}{c}[\psi] \\ \vdots \\ \chi\end{array}}{\chi}$$

Falsum

$$\frac{\bot}{\varphi}$$

Split

$$\frac{\alpha \to \psi \mathbin{\vee\mkern-10mu\vee} \chi}{(\alpha \to \psi) \mathbin{\vee\mkern-10mu\vee} (\alpha \to \chi)}$$

$\neg\neg$ elimination

$$\frac{\neg\neg\alpha}{\alpha}$$

Fig. 4 A sound and complete natural-deduction system for InqB. The variables φ, ψ, χ range over all formulas in $\mathcal{L}$, while the variable α is restricted to classical formulas.

sion from the assumption that we have settled $\varphi \mathbin{\vee\mkern-10mu\vee} \psi$: after all, to settle $\varphi \mathbin{\vee\mkern-10mu\vee} \psi$ amounts precisely to settling one of the disjunct.

Notice that the same does not hold for *classical* disjunction: an information state may well settle a classical disjunction without settling either disjunct. As a consequence, the standard elimination rule is not generally sound for classical disjunction in the inquisitive setting. Indeed, if it were, then classical and inquisitive disjunction would be provably equivalent. Instead, Figure 5 shows some derived rules for classical disjunction in the inquisitive setting. While the standard introduction rule is generally valid, the elimination rule must be restricted to conclusions that are classical formulas. This restriction prevents obviously unsound derivations such as $p \vee \neg p \vdash {?p}$.

The remaining two rules of our derivation system correspond to properties of inquisitive logic that we have discussed in detail. The rule of double negation elimination, which is restricted to classical formulas, captures the fact that such formulas are statements. Indeed as Proposition 19 ensures, the validity of double negation is characteristic of statements, i.e., of formulas which are specific. Notice that, by including double negation elimination for all classical formulas, our system is an extension of a standard natural deduction system for classical logic, as given, e.g., in Gamut (1991).[15]

[15] In fact, allowing $\neg\neg$-elimination for *atoms* would be sufficient to ensure completeness.

Classical disjunction

$$\frac{\varphi}{\varphi \vee \psi} \qquad \frac{\psi}{\varphi \vee \psi} \qquad \frac{\varphi \vee \psi \quad \begin{array}{c}[\varphi] \\ \vdots \\ \alpha\end{array} \quad \begin{array}{c}[\psi] \\ \vdots \\ \alpha\end{array}}{\alpha}$$

Negation

$$\frac{\begin{array}{c}[\varphi] \\ \vdots \\ \bot\end{array}}{\neg\varphi} \qquad \frac{\varphi \quad \neg\varphi}{\bot}$$

Fig. 5 Derived rules for the defined connectives, where α is restricted to classical formulas.

Finally, the role of the split rule is to capture the Split Property discussed above. Recall that Proposition 17 ensures that, if α is a statement, we have:

$$\alpha \models_s \psi \mathbin{\vee\!\!\!\vee} \chi \;\text{ implies }\; \alpha \models_s \psi \text{ or } \alpha \models_s \chi$$

In view of the connection between contextual entailment and implication, and of the support conditions for $\mathbin{\vee\!\!\!\vee}$, this property can be restated as follows: $s \models \alpha \to \psi \mathbin{\vee\!\!\!\vee} \chi$ implies $s \models (\alpha \to \psi) \mathbin{\vee\!\!\!\vee} (\alpha \to \chi)$. In turn, this means that the following entailment is valid in InqB:

$$\alpha \to \psi \mathbin{\vee\!\!\!\vee} \chi \models (\alpha \to \psi) \mathbin{\vee\!\!\!\vee} (\alpha \to \chi)$$

Since any classical formula is a statement, the split rule is a special case of this entailment pattern, which is general enough to ensure that all other instances are provable as well.[16]

4.2 Inquisitive proofs and their constructive content

To see what proofs involving questions look like, let us consider once again our initial example of a dependency, corresponding to the following entailment:[17]

$$\gamma, \;\; ?s_1, \;\; ?s_2, \;\; ?g \;\; \models \;\; ?t$$

where γ stands for the protocol description, $t \leftrightarrow s_2 \vee (s_1 \wedge g)$. Since this entailment is valid, it must be possible to provide a proof for it in our system. Such a proof is displayed below, where steps involving only classical logic have been omitted and denoted by $(\mathsf{C}_1), \ldots, (\mathsf{C}_4)$. We will refer to this proof as P.

[16] The $\mathbin{\vee\!\!\!\vee}$-split rule plays exactly the same role in our system that the Kreisel-Putnam axiom plays in Ciardelli and Roelofsen (2011). In this way, what looked like a fundamental but mysterious ingredient of the logic takes on a clear conceptual significance: it captures that a statement can only logically resolve a question in case it entails some resolution of it.

[17] In order to simplify the proof, the conjunctive question $?s_1 \wedge ?s_2$ has been replaced here by two polar questions, $?s_1$ and $?s_2$. This change is merely cosmetic, and dispensable.

$$
\begin{prooftree}
\AxiomC{$?s_2$}
\AxiomC{γ}
\AxiomC{$[s_2]$}
\RightLabel{(C_1)}
\BinaryInfC{t}
\RightLabel{$(\mathbb{V}\mathsf{i})$}
\UnaryInfC{$?t$}
\AxiomC{$?s_1$}
\AxiomC{γ}
\AxiomC{$[\neg s_1]$}
\AxiomC{$[\neg s_2]$}
\RightLabel{(C_2)}
\TrinaryInfC{$\neg t$}
\RightLabel{$(\mathbb{V}\mathsf{i})$}
\UnaryInfC{$?t$}
\AxiomC{$?g$}
\AxiomC{$\gamma \quad [s_1] \quad [g]$}
\RightLabel{(C_3)}
\UnaryInfC{t}
\RightLabel{$(\mathbb{V}\mathsf{i})$}
\UnaryInfC{$?t$}
\AxiomC{$\gamma \quad [\neg s_2] \quad [\neg g]$}
\RightLabel{(C_4)}
\UnaryInfC{$\neg t$}
\RightLabel{$(\mathbb{V}\mathsf{i})$}
\UnaryInfC{$?t$}
\RightLabel{$(\mathbb{V}\mathsf{e})$}
\TrinaryInfC{$?t$}
\RightLabel{$(\mathbb{V}\mathsf{e})$}
\TrinaryInfC{$?t$}
\RightLabel{$(\mathbb{V}\mathsf{e})$}
\TrinaryInfC{$?t$}
\end{prooftree}
$$

It is instructive to consider what argument is included by this proof. In words, this may be phrased roughly as follows. We are assuming information of type $?s_2$. This means that we have either the information that s_2, or the information that $\neg s_2$. If we have the information that s_2, then by combining this information with γ we can infer t, and so we have some information of type $?t$. On the other hand, if we have the information that $\neg s_2$, we have to rely on having information of type $?s_1$. This means that we have either the information that s_1, or the information that $\neg s_1$. If the information we have is $\neg s_1$, then by combining this with $\neg s_2$ and γ we can infer $\neg t$, and thus we have some information of type t. On the other hand, if the information we have is that s_1, then we have to rely on having information of type $?g$. Again, there are two possibilities: if the information we have is g, then by combining this with s_1 and γ we can infer t, and so we have information of type $?t$; if the information we have is $\neg g$, then by combining this with $\neg s_2$ and γ we can infer $\neg t$, and thus again we have information of type $?t$. So, in any case, under the given assumptions we are assured to have information of type $?t$.[18]

Notice an interesting fact about this proof: the proof does not just *witness* that, given γ, information of type $?s_1$, $?s_2$, and $?g$ yields information of type $?t$: within its structure, it actually encodes how to obtain information of type $?t$ from information of types $?s_1, ?s_2$, and $?g$. This means that, if we replace each of the generic assumptions $?s_1, ?s_2, ?g$ in the proof by a corresponding specific assumption, say $s_1, \neg s_2$, and g respectively, the proof actually describe how to obtain a corresponding piece of information of type $?t$. In other words, the proof actually describes an *algorithm* for computing the dependency at hand.

This is not an accident, but a manifestation of a general fact concerning inquisitive proofs: given a proof P which encodes a dependency, we can always see this proof as encoding a program that computes the dependency.

To state the relevant result concisely, let us write $\overline{\varphi}$ for a sequence $\varphi_1, \ldots, \varphi_n$ of formulas, and $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$ to mean that $\overline{\alpha}$ is a sequence $\alpha_1, \ldots, \alpha_n$ such that $\alpha_i \in \mathcal{R}(\varphi_i)$ for $1 \leq i \leq n$.

Theorem 1 (Existence of a Resolution Algorithm)
Let $P : \overline{\varphi} \vdash \psi$ and let $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$. There is a procedure which, inductively on P, constructs a proof $Q : \overline{\alpha} \vdash \beta$ having as conclusion a resolution $\beta \in \mathcal{R}(\psi)$.

[18] The proof of a dependency does not always have to proceed, as in this case, by "splitting cases". E.g., from μ and $\mu \to \nu$, we can immediately infer ν by modus ponens. We do not need to look at the resolutions of μ and $\mu \to \nu$. This is convenient, since the number of resolutions for $\mu \to \nu$ is exponential in the number of resolutions for μ, and typically large.

Proof. Let us describe how to construct the proof Q inductively on P. We distinguish a number of cases depending on the last rule applied in P.

- $\psi = \varphi_i$ is an undischarged assumption. In this case, a resolution $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$ contains a resolution α_i of $\varphi_i = \psi$, and we have a trivial proof of $\overline{\alpha} \vdash \alpha_i$.
- $\psi = \chi \wedge \xi$ was obtained by ($\wedge$i) from χ and ξ. Then the immediate subproofs of P are a proof $P' : \overline{\varphi} \vdash \chi$ and a proof $P'' : \overline{\varphi} \vdash \xi$. Take any resolution $\overline{\alpha}$ of $\overline{\varphi}$. The induction hypothesis gives us two proofs $Q' : \overline{\alpha} \vdash \beta$ and $Q'' : \overline{\alpha} \vdash \gamma$, where $\beta \in \mathcal{R}(\chi)$ and $\gamma \in \mathcal{R}(\xi)$. By extending these proofs with an application of ($\wedge$i), we get a proof $Q : \overline{\alpha} \vdash \beta \wedge \gamma$, and we are done since $\beta \wedge \gamma \in \mathcal{R}(\chi \wedge \xi)$.
- $\psi = \chi \to \xi$ was obtained by ($\to$i). Then the immediate subproof of P is a proof $P' : \overline{\varphi}, \chi \vdash \xi$. Now take any resolution $\overline{\alpha}$ of $\overline{\varphi}$. Suppose $\beta_1, \dots, \beta_m$ are the resolutions of χ. For $1 \leq i \leq m$, then, the sequence $\overline{\alpha}, \beta_i$ is a resolution of $\overline{\varphi}, \chi$. Thus, by induction hypothesis we have a proof $Q_i' : \overline{\alpha}, \beta_i \vdash \gamma_i$ for some resolution γ_i of ξ. But then, extending Q_i' with an application of ($\to$i), have a proof $Q_i'' : \overline{\alpha} \vdash \beta_i \to \gamma_i$. And since this is the case for any $1 \leq i \leq n$, by a number of ($\wedge$i) rules we obtain a proof $Q : \overline{\alpha} \vdash (\beta_1 \to \gamma_1) \wedge \dots \wedge (\beta_m \to \gamma_m)$ which is what we need, since by construction $(\beta_1 \to \gamma_1) \wedge \dots \wedge (\beta_m \to \gamma_m)$ is a resolution of $\chi \to \xi$.
- $\psi = \chi \vee\!\!\!\vee \xi$ was obtained by ($\vee\!\!\!\vee$i) from one of the disjuncts. Without loss of generality, let us assume it is χ. Thus, the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \chi$. Take any resolution $\overline{\alpha}$ of $\overline{\varphi}$. The induction hypothesis gives us a proof $Q : \alpha \vdash \beta$ for some $\beta \in \mathcal{R}(\chi)$. Since β is also a resolution of $\chi \vee\!\!\!\vee \xi$, we are done: Q itself is the proof we need.
- ψ was obtained by ($\wedge$e) from $\psi \wedge \chi$. Then the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \psi \wedge \chi$. Take a resolution $\overline{\alpha}$ of $\overline{\varphi}$. The induction hypothesis gives a proof $Q' : \overline{\alpha} \vdash \beta$, where $\beta \in \mathcal{R}(\psi \wedge \chi)$. By definition of resolutions for a conjunction, β is of the form $\gamma \wedge \gamma'$ where $\gamma \in \mathcal{R}(\psi)$ and $\gamma' \in \mathcal{R}(\chi)$. Extending Q' with an application of ($\wedge$e) we have a proof $Q : \overline{\alpha} \vdash \gamma$, as we needed. Of course, the argument is analogous if ψ was obtained by ($\wedge$e) from a conjunction $\chi \wedge \psi$.
- ψ was obtained by ($\to$e) from χ and $\chi \to \psi$. Then the immediate subproofs of P are a proof $P' : \overline{\varphi} \vdash \chi$, and a proof $P'' : \overline{\varphi} \vdash \chi \to \psi$. Consider a resolution $\overline{\alpha}$ of $\overline{\varphi}$. The induction hypothesis gives us a proof $Q' : \overline{\alpha} \vdash \beta$ where $\beta \in \mathcal{R}(\chi)$, and a proof $Q'' : \overline{\alpha} \vdash \gamma$, where $\gamma \in \mathcal{R}(\chi \to \psi)$. Now, if $\mathcal{R}(\chi) = \{\beta_1, \dots, \beta_m\}$, then $\beta = \beta_i$ for some i, and by definition of the resolutions of an implication, $\gamma = (\beta_1 \to \gamma_1) \wedge \dots \wedge (\beta_n \to \gamma_m)$ where $\{\gamma_1, \dots, \gamma_n\} \subseteq \mathcal{R}(\psi)$. Now, extending Q'' with an application of ($\wedge$e) we get a proof $Q''' : \overline{\alpha} \vdash \beta_i \to \gamma_i$. Putting together this proof with Q' and applying ($\to$e), we obtain $Q : \overline{\alpha} \vdash \gamma_i$, whose conclusion is a resolution of ψ.
- ψ was obtained by ($\vee\!\!\!\vee$e) from $\chi \vee\!\!\!\vee \xi$. Then the immediate subproofs of P are: a proof $P' : \overline{\varphi} \vdash \chi \vee\!\!\!\vee \xi$; a proof $P'' : \overline{\varphi}, \chi \vdash \psi$; and a proof $P''' : \overline{\varphi}, \xi \vdash \psi$. Take a resolution $\overline{\alpha}$ of $\overline{\varphi}$. The induction hypothesis applied to P' gives us a proof $Q' : \overline{\alpha} \vdash \beta$ for some $\beta \in \mathcal{R}(\chi \vee\!\!\!\vee \xi) = \mathcal{R}(\chi) \cup \mathcal{R}(\xi)$. Without loss of generality, assume that $\beta \in \mathcal{R}(\chi)$. Then the sequence $\overline{\alpha}, \beta$ is a resolution

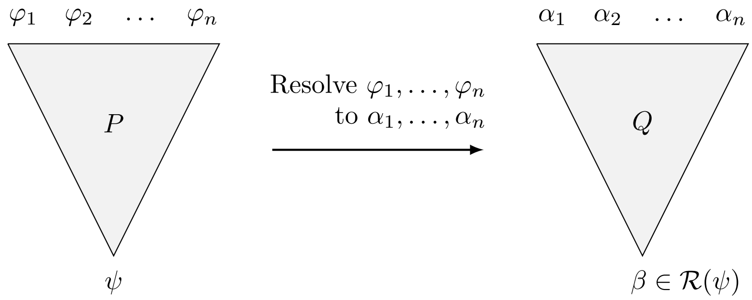


Fig. 6 An illustration of the resolution algorithm: given a proof $P : \overline{\varphi} \vdash \psi$ and resolutions $\overline{\alpha}$ of $\overline{\varphi}$, the algorithm builds a proof $Q : \overline{\alpha} \vdash \beta$ of a corresponding resolution β of ψ.

of $\overline{\varphi}, \chi$. Thus, the induction hypothesis applied to P'' gives us a proof $Q'' : \overline{\alpha}, \beta \vdash \gamma$ for some $\gamma \in \mathcal{R}(\psi)$. Now, by substituting any undischarged assumption of β in Q'' by an occurrence of Q', we obtain a proof $Q : \overline{\alpha} \vdash \gamma$ having a resolution of ψ as its conclusion, as we wanted.

- ψ was obtained by ($\bot$e). This means that the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \bot$. Take any resolution $\overline{\alpha}$ of $\overline{\varphi}$. Since $\mathcal{R}(\bot) = \{\bot\}$, the induction hypothesis gives a proof $Q' : \overline{\alpha} \vdash \bot$. By extending Q' with an application of ($\bot$e), we then have a proof $Q : \{\alpha_1, \dots, \alpha_n\} \vdash \beta$ which concludes an arbitrary formula β, in particular a resolution of ψ (notice that, by definition, the set of resolutions of a formula is always non-empty).
- $\psi = (\alpha \to \chi) \mathbin{\vee\!\!\vee} (\alpha \to \xi)$ was obtained by an application of the $\mathbin{\vee\!\!\vee}$-split rule from $\alpha \to \chi \mathbin{\vee\!\!\vee} \xi$, where $\alpha \in \mathcal{L}_c$. Then, the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \alpha \to \chi \mathbin{\vee\!\!\vee} \xi$. Take a resolution $\overline{\beta}$ of $\overline{\varphi}$. The induction hypothesis gives us a proof $Q : \overline{\beta} \vdash \gamma$ where $\gamma \in \mathcal{R}(\alpha \to \chi \mathbin{\vee\!\!\vee} \xi)$. Now, we know from Proposition 11 that $\mathcal{R}(\alpha) = \{\alpha\}$. Thus, the formula γ must be of the form $\alpha \to \delta$, where $\delta \in \mathcal{R}(\chi \mathbin{\vee\!\!\vee} \xi) = \mathcal{R}(\chi) \cup \mathcal{R}(\xi)$. Without loss of generality, assume that $\delta \in \mathcal{R}(\chi)$: then $\gamma = \gamma \to \delta$ is also a resolution of $\alpha \to \chi$ and thus also a resolution of $(\alpha \to \chi) \mathbin{\vee\!\!\vee} (\alpha \to \xi)$. Hence, Q itself is the proof we need.
- $\alpha \in \mathcal{L}_c$ was obtained by double negation elimination from $\neg\neg\alpha$. In this case, the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \neg\neg\alpha$. Take any resolution $\overline{\beta}$ of $\overline{\varphi}$. Now, since $\neg\neg\alpha$ is a classical formula, by Proposition 11 we have $\mathcal{R}(\neg\neg\alpha) = \{\neg\neg\alpha\}$. Thus, the induction hypothesis gives a proof $Q' : \overline{\beta} \vdash \neg\neg\alpha$. Extending Q' with an application of double negation elimination we obtain a proof $Q : \overline{\beta} \vdash \alpha$, which is what we need, since α is a classical formula and thus a resolution of itself.

□

We will refer to the inductive procedure described in the previous proof as the *resolution algorithm*. The existence of this procedure shows that an inquisitive proof may be regarded as a function which takes as input specific resolutions of the assumptions, and outputs a proof in classical logic.

For instance, suppose we get the data relative to a certain patient: this patient has only symptom S_1 and is in good physical condition. Then we can instantiate the question assumptions of our proof to $s_1, \neg s_2$, g. Applying the resolution algorithm yields the following proof, witnessing that this particular resolution of the assumptions determines the resolution t to the question $?t$.

$$\frac{\gamma \quad s_1 \quad g}{t}\ (\mathsf{C}_3)$$

This illustrates the insight we discussed above: in a proof, questions may be seen as placeholders for arbitrary information of the corresponding type. A proof containing such placeholders may be seen as a *template* for classical proofs: whenever we are given some specific resolution of the assumptions—say, whenever we get the data relative to a specific patient—the template can be instantiated to a classical proof which infers some specific resolution of the conclusion—in our example, a specific deliberation about the treatment.

5 Relation with previous work

5.1 Non-entailment directed approaches to questions

Throughout most of the history of logic, virtually no attention has been paid to questions. It is not until the second half of the 20th century that logical works devoted to questions have started to appear. In most of these works (e.g. Åqvist, 1965; Harrah, 1961, 1963; Belnap and Steel, 1976; Tichy, 1978) the emphasis has been on providing a logical language for questions, and on characterizing the relation of *answerhood* between statements and questions. Other approaches have focused instead on the role of questions in processes of inquiry, either modeling inquiry itself as a sequence of questioning moves and inference moves, as in the *interrogative model of inquiry* of Hintikka (1999), or characterizing how questions are arrived at in an inquiry scenario, as in the *inferential erotetic logic* of Wiśniewski (1995).[19] What all these theories have is common is the assumption that dealing with questions requires turning to relations other than logical entailment. Thus, they pursue enterprises which, while related, are also different in an important respect from the one that we have been concerned with here: incorporating questions on a par with statements in the very relation of entailment, and characterizing how they can be manipulated in entailment-tracking logical proofs.

5.2 The Logic of Interrogation

To the best of our knowledge, the first approach that allows for a generalization of the classical notion of entailment to questions is the Logic of Interrogation (LoI) of Groenendijk (1999), based on the partition theory of questions of Groenendijk and Stokhof (1984). The original presentation of the system is

[19] For some discussion on the relations between inferential erotetic logic and inquisitive semantics, see Wiśniewski and Leszczyńska-Jasion (2015) and Ciardelli et al. (2015).

a dynamic one, in which entailment is defined in terms of context-change potential. However, as pointed out by ten Cate and Shan (2007), the dynamic coating is not essential. In its essence, the system may be described as follows: both statements and questions are interpreted with respect to pairs $\langle w, w' \rangle$ of possible worlds: a statement is satisfied by such a pair if it is true at both worlds, while a question is satisfied if the true answer to the question is the same in both worlds. In this approach, the meaning of a sentence φ is captured by the set of pairs $\langle w, w' \rangle$ satisfying φ; for any φ, this set is an equivalence relation over a subset of the logical space, which we will denote as $\sim_\varphi$. Such an equivalence relation may be equivalently regarded as a partition Π_φ of a subset of the logical space, where the blocks of the partitions are the equivalence classes $[w]^{\sim_\varphi}$ of worlds modulo $\sim_\varphi$.

$$\Pi_\varphi = \{[w]^{\sim_\varphi} \mid w \in \omega\}$$

For a statement α, the partition Π_α always consists of a unique block, corresponding to the truth-set $|\alpha|$ of the statement. For a question μ, Π_μ consists of several blocks, which are regarded as the complete answers to the question.

Since statements and questions are interpreted by means of a uniform semantics, LoI allows for the definition of a notion of entailment in which both statements and questions can take part:

$$\varphi \models_{\mathsf{LoI}} \psi \iff \text{for all } w, w' \in \omega : \langle w, w' \rangle \models \varphi \text{ implies } \langle w, w' \rangle \models \psi$$

In terms of partitions, this notion of entailment may be cast as follows:

$$\varphi \models_{\mathsf{LoI}} \psi \iff \text{for all } a \in \Pi_\varphi \text{ there is } a' \in \Pi_\psi \text{ such that } a \subseteq a'$$

This shows that in LoI, too, we may view sentences as denoting information types; φ entails ψ in case information of type φ always yields information of type ψ. Thus, while this has not been highlighted much in the literature, the unified view of entailment discussed in this paper already emerges in LoI.

In Groenendijk (1999), this approach is applied to a particular logical language, which is an extension of first-order predicate logic with questions. This gives rise to an interesting combined logic of statements and questions, which was investigated and axiomatized by ten Cate and Shan (2007).

The relation between the LoI framework and the approach presented here can be characterized as follows. If a sentence φ is interpretable in LoI, then a state s supports φ in case s is included in one of the blocks of the partition induced by φ. So, the support-set of φ may be obtained as the downward closure of the partition Π_φ.

$$[\varphi] = (\Pi_\varphi)^{\downarrow}$$

Conversely, the elements of the partition Π_φ can always be characterized as the maximal elements of $(\Pi_\varphi)^{\downarrow}$. This means that the LoI-representation of a sentence φ can be recovered from its inquisitive representation as follows:

$$\Pi_\varphi = \textsc{Alt}[\varphi]$$

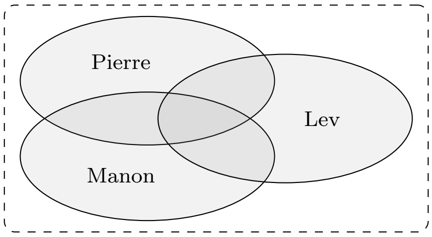


Fig. 7 The alternatives for the mention-some question (3-a), where we have restricted to a small set of candidates {Pierre, Manon, Lev}.

Thus, for sentences that can be interpreted in LoI, we can go back and forth between the two semantics. Furthermore, it is easy to see that the notion of entailment that the two frameworks characterize is the same.

$$\varphi \models \psi \iff \varphi \models_{\mathsf{LoI}} \psi$$

In spite of this tight connection, however, what we have done here is not merely to provide an alternative semantics for LoI, based on information states rather than pairs of worlds. The reason is that the support approach that we discussed in this paper is strictly more general than the LoI approach based on pairs of worlds. To see why, consider again the way in which a question μ is interpreted in LoI: a pair of worlds $\langle w, w' \rangle$ satisfies μ in case *the complete answer* to μ is the same in w as in w'. Clearly, this interpretation only makes sense provided that for any world w, there is such a thing as *the complete answer* to μ at w. Now, our analysis of the relation between complete answers and support conditions makes clear what this assumption amounts to: μ must be a *partition question*, in the following sense.

Definition 8 (Partition questions)
μ is a partition question if any world is contained in a unique alternative for μ.

While the class of partition questions includes many natural kinds of questions, such as the questions that were at play in our example, there are also important types of questions that fall outside of this class. Most importantly, this class does not include so-called *mention-some* questions, that is, questions that ask for an instance of a certain property or relation. Under their most salient interpretation, the following are all examples of mention-some questions:

(3) a. What is a typical French name?
b. Where can I buy an Italian newspaper?
c. How can I get to the station from here?
d. Who has a van that we could borrow?

It is easy to see that such questions are not partition questions. Consider for example (3-a): as illustrated by Figure 7, the alternatives for this question correspond to the possible witnesses for the property of being a typical French name. In a given world w, we may of course have several witnesses for this property, which means that w is contained in several alternatives for the question. This shows that a question like (3-a) is not a partition question, and thus it is not interpretable in LoI. On the other hand, since it is clear what information is needed in order for (3-a) to be settled, the inquisitive approach has no problem interpreting this question. Since mention-some questions are a broad and practically relevant class of questions, this is a significant advantage of inquisitive semantics over the LoI approach.

Another important class of non-partition questions that we briefly discussed in this paper is given by *conditional questions*, exemplified by (4).

(4) If Mary invites you to the party, will you go?

Such questions, too, cannot be adequately represented in the LoI framework. Starting precisely with the problem of conditional questions, the pursuit of greater generality lead Velissaratou (2000), Groenendijk (2009) and Mascarenhas (2009) to relax the constraints of the LoI framework, interpreting sentences by means of binary relations that are not necessarily transitive. This lead to a first version of inquisitive semantics, now referred to as *pair semantics*. While Groenendijk (2011) showed that the pair semantics can indeed deal adequately with conditional questions, Ciardelli (2008, 2009), and later Ciardelli et al. (2015) argued that no pair semantics provides a satisfactory general framework for questions, and that an interpretation based on information states is needed instead, leading to the support-based approach that we discussed here.

5.3 Nelken and Shan's modal approach

After the Logic of Interrogation, a different uniform approach to statements and questions was proposed by Nelken and Shan (2006). In this approach, questions are translated as modal sentences, and they are interpreted by means of truth-conditions: a question is true at a world w in case it is settled by an information state $R[w]$ associated with the world (i.e., the set of successors given by an accessibility relation R). Thus, for instance, Nelken and Shan render the question *whether* p by the modal formula $?p := \Box p \vee \Box\neg p$.

In one respect, this approach is similar to the approach proposed in this paper, since the meaning of a question is taken to be encoded by the conditions under which the question is settled by a relevant body of information. And indeed, if we consider entailments which involve only questions, the approach of Nelken and Shan makes the expected predictions. However, an asymmetry between statements and questions is maintained in this approach: for questions, what matters is whether they are settled by a relevant information state, while for statements, what matters is whether they are true at the world of evaluation. This asymmetry creates problems the moment we start considering cases of entailment involving both statements and questions, such as the one corresponding to our protocol example. It is easy to see that, if such entailments are to be meaningful at all, entailment cannot just amount to preservation of truth. Nelken and Shan propose to fix this by re-defining entailment as *modal* *consequence*: $\varphi \models \psi$ if, whenever φ is true at *every* possible world in a model, so is ψ. However, this move has the odd consequence of changing the consequence relation for statements in an undesirable way. For instance, if our declarative language indeed contains a Kripke modality, say a knowledge modality K, then if our notion of entailment is redefined as modal consequence, we make undesirable predictions, such as $p \models Kp$. Thus, this approach does not really allow us to extend classical logic with questions in a conservative way.[20]

5.4 The modal translation of InqB

The asymmetry between statements and questions that is problematic for Nelken and Shan's approach can be eliminated by letting statements, too, be interpreted in terms of when they are settled by the state $R[w]$, rather in terms when they are true at w. That is, just like Nelken and Shan translate a question $?p \in \mathcal{L}$ as $\Box p \vee \Box\neg p$, one may translate a statement $p \in \mathcal{L}$ in modal logic as $\Box p$. More generally, we may associate to any formula $\varphi \in \mathcal{L}$ a modal formula $\varphi^\Box$ defined as follows:

$$\varphi^\Box = \bigvee\{\Box\alpha \mid \alpha \in \mathcal{R}(\varphi)\}$$

We will refer to $\varphi^\Box$ as the *modal translation* of φ. It is then easy to show that this map is indeed a translation of InqB into the minimal normal modal logic K, in the sense that for any $\Phi \cup \{\psi\} \subseteq \mathcal{L}$, we have:

$$\Phi \models \psi \iff \Phi^\Box \models_{\mathsf{K}} \psi^\Box$$

where $\Phi^\Box = \{\varphi^\Box \mid \varphi \in \Phi\}$. This translation shows that it is possible to encode InqB within the modal logic K. We could compare this with the Gödel translation of intuitionistic logic into the modal logic S4. Thus, modal logic provides an alternative setup in which dependencies may be captured. E.g., instead of writing $p \leftrightarrow q,\ ?p \models ?q$, we may write $\Box(p \leftrightarrow q),\ \Box p \vee \Box\neg p \models_{\mathsf{K}} \Box q \vee \Box\neg q$. However, the approach presented in this paper has several important advantages over the modal approach. Let us discuss some of them.

Parsimony. First of all, the modal approach is unnecessarily redundant. In this approach, a formula is evaluated at a world w equipped with an information state $R[w]$. However, it is clear that the evaluation world w is completely unnecessary: it is only the content of the information state $R[w]$ that matters for the satisfaction of a formula $\varphi^\Box$. But if it is only the content of a certain

[20] Besides this, there are other difficulties, too. First, it is hard to make sense of the truth-conditions for questions. For instance, is the question *what is the capital of Spain* true at the actual world? The truth-conditions of a sentence depend on a given information state, but it is not clear what particular information state we should consider in assessing the truth-conditions of the question at a world. Second, while in Nelken and Shan's system questions can be embedded under logical operations, we do not always get the right results. For instance, the material conditional $p \to ?q$, that is, $\neg p \vee (\Box q \vee \neg\Box q)$, is not a correct rendering of a conditional question, such as (2). Such a question does not ask for a resolution of $?q$ *if p happens to be true*, but for a resolution of $?q$ *under the assumption that p*.

information state that matters, we can evaluate a formula directly with respect to that information state, without invoking a specific world of evaluation. This move allows for a significant simplification of our semantic structures: we no longer need an accessibility relation R, whose unique purpose was to anchor the relevant state to a specific world.

Insight. By uncovering the connection between dependency and entailment, the present approach provides an insight that is missing in the modal approach. This insight also has practical consequences, since it allows us to use ideas and techniques of logic in the analysis of the dependencies. For instance, we saw that, since entailment can be generally internalized in the language by means of implication, dependencies can be expressed as implications between questions. This provides a well-behaved logical representation of dependencies, and suggests natural rules for reasoning with them. As an example of the explanatory power of the approach, this perspective shows that the well-known Armstrong's axioms for dependency used in database theory are nothing but the familiar intuitionistic rules for implication in disguise (a point first made by Abramsky and Väänänen, 2009).

Logical operations. In intuitionistic logic there is a wealth of interesting structure that becomes rather invisible from the standpoint of the $\mathsf{S4}$ translation. Similarly, we have seen that the inquisitive approach leads to the discovery of interesting structural features at the support level: in particular, connectives such as conjunction and implication (as well as quantifiers and modalities, which we have not discussed here) generalize to this setting in a natural way, so that they can also manipulate questions. These generalizations do not only give nice results: they are also natural from an algebraic point of view Roelofsen (2013) and from a proof-theoretic one, as we saw. To give just one example, let us focus on the implication operation. Consider the following sentences:

(5)
a. Alice will come to the party.
b. Bob will come to the party.
c. Will Bob come to the party?
d. If Alice comes to the party, Bob will come to the party.
e. If Alice comes to the party, will Bob come to the party?

In inquisitive logic, the interpretation of these sentences can be obtained in a simple, compositional way. If we translate (5-a) as p and (5-b) as q, then we can translate (5-c) as $?q$, (5-d) as $p \to q$, and (5-e) as $p \to ?q$. Notice that one and the same operator is at play in (5-d) and (5-e): this operator has a uniform semantics, a simple algebraic characterization, and a natural proof-theory.

In the modal approach, the translation of (5-a) is $\Box p$, and the translation of (5-b) is $\Box q$. The translation of (5-c) is $\Box p \vee \Box\neg p$. The translation of (5-d) is not $\Box p \to \Box q$, as would be expected, but rather $\Box(p \to q)$; similarly, the translation of (5-e) is not $\Box p \to (\Box q \vee \Box\neg q)$, but rather $\Box(p \to q) \vee \Box(p \to \neg q)$.

Although modal logic does have an implication, this cannot be used to interpret the conditional construction. More importantly, in this approach, it is not clear that there is any structural similarity between (5-d) and (5-e), nor that there is a fundamental relation between these two sentences and the simpler sentences (5-a-c). Clearly, some important piece of structure—in the case in point, the existence of a neat implication operation—is being missed from the modal perspective.

Inferences and computational interpretation of proofs. We saw in Section 4 that inquisitive logic allows us to manipulate questions in inference by means of simple and familiar logical rules. By using questions, we can then provide formal proofs of dependencies. Moreover, we saw that such proofs have a computational interpretation, encoding algorithms to compute the dependency at hand. It is not clear that the modal approach has an equally attractive framework to offer. First, it would require us to reason with a more complex language, including modalities in addition to just connectives (or, in addition to whatever other constants the language includes). Second, it is not clear whether a general constructive interpretation of proofs exists in this approach.

5.5 Dependence logic

The ideas discussed in this chapter are also deeply connected with the investigations undertaken in recent years within the framework of Dependence Logic (Väänänen, 2007). Indeed, the dependencies considered in dependence logic are special instances of question entailment. In particular, the dependence atoms $=(p_1, \dots, p_n, q)$ of *propositional dependence logic* (Väänänen, 2008; Yang, 2014) capture the dependency of the atomic polar question $?q$ on the atomic polar questions $?p_1, \dots, ?p_n$. As a consequence, in our language they may be expressed as $?p_1 \wedge \dots \wedge ?p_n \to ?q$. This yields a decomposition of these atoms into more basic and better-behaved operations—which allows for a natural proof-theory. While the possibility of such a decomposition was noted by Abramsky and Väänänen (2009), the present work casts new light on this connection in several ways. First, we can now see that this decomposition reflects a fundamental connection between dependencies and questions: a dependency is a case of entailment having questions as its protagonists; since entailments can be internalized as implications, dependencies can be expressed as implications between questions. Second, it becomes clear that dependence atoms capture only a special case of a more general phenomenon: dependencies may involve all sort of questions other than atomic polar questions, and implication gives us a fully general way to express them. Third, as we have seen, the connection between questions and dependencies has a proof-theoretic side to it: dependencies may be formally proved to hold in a system equipped with questions; and moreover, the resulting proofs do not just *witness* dependencies but actually *encode* dependencies.

The relation between inquisitive logic and dependence logic is the subject of a separate paper (Author, 2015), which develops these points in detail, and shows how they carry over to the setting of first-order logic.

5.6 Previous work on inquisitive semantics

Finally, within the landscape of recent work on inquisitive semantics (among others, Ciardelli et al., 2013, 2015; Roelofsen, 2013; Ciardelli and Roelofsen, 2015; Groenendijk and Roelofsen, 2015; Punčochář, 2015a,b), the contribution of the present paper is threefold. First, we have shown that propositional inquisitive logic may be regarded as a conservative extension of classical logic with a question operator, and we saw that this perspective sheds new light on several logical features of the system. Second, we have investigated the role of questions in logical inferences, and we have established a result, Theorem 1, which brings out the computational content of inquisitive proofs. Thirdly, and most importantly, while work on inquisitive semantics has so far been mainly driven by motivations stemming from linguistics and philosophy of language, we have argued that the move from a classical semantics to an inquisitive semantics also has solid motivations within the field of logic. As we saw, taking questions into account broadens the scope of classical logic in an exciting way, bringing within reach an elegant account of the relation of dependency.

6 Conclusion and further work

In this paper we have seen that, by moving from a truth-based view of meaning to a support-based view, we obtain a uniform semantic framework for statements and questions. This allows for a generalization of the classical logical notions. In particular, while classical logic is concerned with the relation of logical consequence, which relates specific pieces of information, the presence of question allows us to also capture also the relation of logical dependency, which connects different information *types*, and which plays an important role in a broad range of different contexts.

Additionally, we saw that questions have a role to play in logical proofs. By using questions, we can manipulate *generic information*, that is, information which is not fully specified, such as *what the symptoms are* or *whether the treatment is prescribed*. This allows us to formally prove that a certain dependency holds within a certain context. Moreover, we saw that (at least in the propositional setting) from such a proof we can extract an algorithm that computes this dependency, i.e., that turns information of the type described by the assumptions into information of the type described by the conclusion.

In this paper, we focused on the fundamental ideas of the inquisitive approach, and on the generalized view of logic to which they give rise. Now that the significance of inquisitive logic and its relevance for applications become clearer, however, many technical questions also become more urgent. One class of questions is proof-theoretical: e.g., does the natural-deduction system introduced above admit a *normalization theorem*? If not, can inquisitive logic be regimented in a better-behaved proof system? The labeled sequent calculus developed by Sano (2009) for a related logic provides a starting point for an alternative approach. Other interesting questions concern the computational properties of the resolution algorithm, as well as the complexity of the problem of deciding if a given entailment holds in InqB.

A further question is whether an interpretation of the kind discussed here is available for the extension of InqB proposed and axiomatized by Punčochář (2015b). In this extension, the system is expanded with operators that yield non-persistent meanings. The resulting language includes not only formulas like p and $?p$, which are supported if certain information is available, but also formulas like $\sim p$, which are supported if certain information is *lacking*. Understanding the role of such formulas in a logic of information and the properties of the resulting system is an interesting task for future work.

Finally, and most importantly, propositional logic is only a starting point towards a comprehensive theory of questions in logic. From the present perspective, an important result would be an axiomatization of first-order inquisitive logic (Ciardelli, 2009; Ciardelli et al., 2012), ideally paired with a resolution algorithm allowing us to regard proofs as programs for computing dependencies. Within a first-order language, many interesting kinds of questions become expressible, besides the disjunctive questions built up by means of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$. This includes mention-all *wh*-questions like (6-a), which can be expressed as $\forall x?Px$, and mention-some *wh*-questions like (6-b), which can be expressed as $\overline{\exists}xPx$, where $\overline{\exists}$ is the quantifier counterpart of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$.

(6) a. What are the planets in the Solar system?
b. What is one planet in the Solar system?

An axiomatization of inquisitive predicate logic, or a suitable fragment thereof, would provide the means to reason about dependencies between such questions, thus covering a broad spectrum of interesting informational scenarios.

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