Interrogative dependencies and the constructive content of inquisitive proofs

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Abstract. This paper shows how dichotomous inquisitive semantics gives rise to a general notion of entailment that unifies standard declarative entailment with answerhood and interrogative dependency, the relation holding when an answer to a question determines an answer to another. We investigate the associated logic, presenting a new completeness proof based on an explicit canonical model construction. On the way to this proof, we establish a new result, the *resolution theorem*, which shows that inquisitive proofs have a natural computational interpretation. We conclude arguing that, as a logic of dependencies, inquisitive logic has certain theoretical and practical advantages over related systems.

Keywords: Inquisitive logic, logic of questions, dependence logic.

1 Introduction

Inquisitive semantics [9, 1, 14, 4, a.o.] pursues a semantic framework that encompasses both information and issues, thus reflecting the primary function of language as a tool for information exchange. The most standard logical incarnation of the framework is *basic inquisitive semantics*, InqB [1, 5], obtained associating the connectives with the natural algebraic operations in the space of inquisitive meanings [14]. Recently, a close relative of InqB has been investigated, the system InqD_π of *dichotomous inquisitive semantics* [10, 3], whose syntax enforces a strict distinction of formulas into declaratives and interrogatives.

In the present paper, we take a closer look at the logic that arises from this system. We point out that this logic subsumes standard declarative entailment, answerhood, and interrogative dependencies as three particular cases of a unique, cross-categorial entailment relation. We provide a new completeness proof, more explicit and better suited to generalizations than the one given in [3]. On our way to this proof, we establish a new result which brings out how inquisitive proofs may be seen as encoding methods for computing interrogative dependencies. Finally, we look at InqD_π from the perspective of dependence logic, pointing out certain advantages of the inquisitive logical setup.

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2 Dichotomous inquisitive semantics

This section provides a minimal introduction to dichotomous inquisitive semantics. For a comprehensive exposition and proofs, the reader is referred to [3].

Unlike *basic* inquisitive semantics InqB, which is based on a standard propositional language, *dichotomous* inquisitive semantics InqD_π enriches a propositional language, whose formulas are called *declaratives*, with a new syntactic category of *interrogative* formulas. Given a set of atoms $\mathcal{P}$, the set $\mathcal{L}_!$ of declaratives and the set $\mathcal{L}_?$ of interrogatives are defined recursively as follows:[1]

Definition 1 (Syntax).

1. *for any* $p \in \mathcal{P}$, $p \in \mathcal{L}_!$
2. $\bot \in \mathcal{L}_!$
3. *if* $\alpha_1, \ldots, \alpha_n \in \mathcal{L}_!$, *then* $?\{\alpha_1, \ldots, \alpha_n\} \in \mathcal{L}_?$
4. *if* $\varphi, \psi \in \mathcal{L}_\circ$, *then* $\varphi \wedge \psi \in \mathcal{L}_\circ$, *where* $\circ \in \{!, ?\}$
5. *if* $\varphi \in \mathcal{L}_! \cup \mathcal{L}_?$ *and* $\psi \in \mathcal{L}_\circ$, *then* $\varphi \to \psi \in \mathcal{L}_\circ$, *where* $\circ \in \{!, ?\}$

We will also make use of some abbreviations. We will write $\neg\varphi$ for $\varphi \to \bot$ and $\varphi \leftrightarrow \psi$ for $(\varphi \to \psi) \wedge (\psi \to \varphi)$. Moreover, for α and β declaratives, we will write $\alpha \vee \beta$ for $\neg(\neg\alpha \wedge \neg\beta)$, and $?\alpha$ for a *polar interrogative* $?\{\alpha, \neg\alpha\}$.

Throughout the paper, we let α, β, γ range over declaratives, μ, ν, λ over interrogatives, and φ, ψ, χ over the whole language. Moreover, Γ ranges over sets of declaratives, Λ over sets of interrogatives, and Φ over arbitrary sets.

The semantics is based on possible world models for propositional logic.[2]

Definition 2 (Models). *A model for* InqD_π *is a pair* $M = \langle \mathcal{W}, V \rangle$, *where:*

- $\mathcal{W}$ *is a set whose elements we refer to as* possible worlds*;*
- $V : \mathcal{W} \to \wp(\mathcal{P})$ *is a* valuation map*, yielding for each* $w \in \mathcal{W}$ *the set* $V(w)$ *of atoms true at* w*.*

Usually, semantics is synonymous with truth-conditions. However, our language now contains interrogatives as well. We do not lay out the meaning of an interrogative by specifying what a state of affairs has to be like to make it true, but rather by specifying what information is needed to resolve it. Thus, the natural evaluation points for interrogatives are not possible worlds, but rather bodies of information. These as referred to as *information states* and modeled formally by identifying them with the set of worlds compatible with the information.

Definition 3 (Information states).
An information state *in a model* M *is a set* $s \subseteq \mathcal{W}$ *of possible worlds.*

[1] Our language is richer than the one in [3], which does not allow interrogatives as antecedents of an implication. This enrichment is essential: as we will see, interrogative dependencies are expressed precisely by implications among two interrogatives.

[2] In [3], a fixed model ω is assumed, consisting of *all* propositional valuations. Since any possible world model is embeddable in ω, this difference is not an essential one.

To retain uniform semantic notions, InqD_π lifts the interpretation of *all* sentences in the language to the level of information states. The semantics is thus given in the form of a relation of *support* between information states and formulas. Intuitively, for a declarative being supported in a state s amounts to being *established* in s, while for an interrogative it amounts to being *resolved* in s.

Definition 4 (Support). *Let M be a model and s an information state in M.*

1. $M, s \models p \iff p \in V(w)$ *for all worlds* $w \in s$
2. $M, s \models \bot \iff s = \emptyset$
3. $M, s \models ?\{\alpha_1, \dots, \alpha_n\} \iff M, s \models \alpha_1$ *or ... or* $M, s \models \alpha_n$
4. $M, s \models \varphi \wedge \psi \iff M, s \models \varphi$ *and* $M, s \models \psi$
5. $M, s \models \varphi \to \psi \iff$ *for any* $t \subseteq s$, *if* $M, t \models \varphi$ *then* $M, t \models \psi$

A first, crucial feature of the semantics is that support is *persistent*.

Fact 1 (Persistence) *If $M, s \models \varphi$ and $t \subseteq s$, then $M, t \models \varphi$.*

Second, although our semantics is defined in terms of support, we can recover *truth* at worlds by defining it as support at the corresponding singleton state.

Definition 5 (Truth).
φ is true *at a world w in M, notation $M, w \models \varphi$, in case $M, \{w\} \models \varphi$.*

Computing the support clauses for singleton states, we find that the connectives all get their standard truth-conditional clauses. Moreover, persistence implies that a world makes a formula true iff it is contained in some supporting state.

Fact 2 $M, w \models \varphi \iff w \in s$ *for some state s such that* $M, s \models \varphi$.

In general, truth conditions do not determine support conditions. For instance, the polar interrogatives $?p$ and $?q$ are both true everywhere, but clearly, in general they have different support conditions. However, the semantics of *declaratives* is still completely determined by truth-conditions: for, a declarative is supported in a state iff it is true at all the worlds in the state.

Fact 3 *For any declarative α:* $M, s \models \alpha \iff (M, w \models \alpha$ *for all* $w \in s)$

To any formula φ, we associate a set $\mathcal{R}(\varphi)$ of declaratives that we call the *resolutions* of φ. This set is defined recursively as follows.

Definition 6 (Resolutions).

- $\mathcal{R}(p) = \{p\}$
- $\mathcal{R}(\bot) = \{\bot\}$
- $\mathcal{R}(?\{\alpha_1, \dots, \alpha_n\}) = \{\alpha_1, \dots, \alpha_n\}$
- $\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\varphi)$ *and* $\beta \in \mathcal{R}(\psi)\}$
- $\mathcal{R}(\varphi \to \psi) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)} \alpha \to f(\alpha) \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$

It is easy to see that a declarative α has itself as unique resolution: $\mathcal{R}(\alpha) = \{\alpha\}$. As for an interrogative μ, we may think of its resolutions of as syntactically generated answers to it.[3] Indeed, the next fact says that establishing some resolution is a necessary and sufficient condition to resolve an interrogative.

Fact 4 *For any M, s and φ,* $M, s \models \varphi \iff M, s \models \alpha$ *for some* $\alpha \in \mathcal{R}(\varphi)$

As a corollary, we get the following normal form result: every formula φ is equivalent to a basic interrogative having the resolutions of φ as constituents.

Corollary 1 (Normal form). *For any φ,* $\varphi \equiv ?\mathcal{R}(\varphi)$.

In terms of resolutions we define the notion of *presupposition* of an interrogative.

Definition 7 (Presupposition of an interrogative).
The presupposition of an interrogative μ is the declarative $\pi_\mu = \bigvee \mathcal{R}(\mu)$.

Since the interrogative operator has the same truth conditions as a disjunction, it follows from Corollary 1 that μ and π_μ have the same truth conditions. Incidentally, this tells us how truth should be read for interrogatives: an interrogative μ is true in w iff some resolution to μ is true in w.

The notion of resolution may be generalized to sets of formulas as follows.

Definition 8 (Resolutions of a set).
The set $\mathcal{R}(\Phi)$ of resolutions of a set Φ contains those sets Γ of declaratives s.t.:

- *for all $\varphi \in \Phi$ there is an $\alpha \in \Gamma$ such that $\alpha \in \mathcal{R}(\varphi)$*
- *for all $\alpha \in \Gamma$ there is a $\varphi \in \Phi$ such that $\alpha \in \mathcal{R}(\varphi)$*

That is, a resolution of Φ is a set of declaratives which is obtained by replacing every formula in Φ by one or more of its resolutions. Since a declarative has itself as unique resolution, we obtain a resolution of Φ by keeping all the declaratives in Φ, and substituting each interrogative by one or more resolutions. Fact 4 generalizes to sets: writing $M, s \models \Phi$ for '$M, s \models \varphi$ for all $\varphi \in \Phi$', we have:

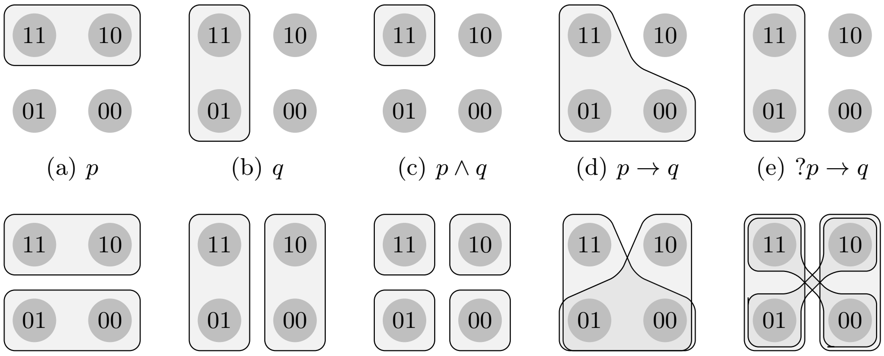
Fact 5 *For any M, s and Φ,* $M, s \models \Phi \iff M, s \models \Gamma$ *for some* $\Gamma \in \mathcal{R}(\Phi)$

We end this section by taking a closer look at the behavior of implication. First, if the antecedent is declarative, the clause amounts to the following simpler one:

$$M, s \models \alpha \to \varphi \quad \iff \quad M, s \cap \{w \in \mathcal{W} \mid M, w \models \alpha\} \models \varphi$$

That is, the conditional $\alpha \to \varphi$ is established (resolved) in s iff φ is established (resolved) in the state that results from augmenting s with the assumption that α is true. For conditional *declaratives*, this delivers a standard material implication, as predicted by Fact 3 together with the fact that truth-conditions are standard.

[3] Our *resolutions* are a more general version of the *basic answers* in the interrogative frameworks of Hintikka [11, 12] and Wisniewski [16]. We use the term *resolutions* as a reminder that this is a specific technical notion, sufficient for the present purposes. Our notion of *presupposition* of a question is also shared with the mentioned theories.



(a) p (b) q (c) $p \wedge q$ (d) $p \to q$ (e) $?p \to q$

(f) $?p$ (g) $?q$ (h) $?p \wedge ?q$ (i) $p \to ?q$ (j) $?p \to ?q$

Fig. 1. The meanings of some simple sentences. 11 represents a world where both p and q are true, 10 a world where p is true and q is false, etc. For simplicity, only *maximal* supporting states are depicted. Notice that, by Fact 3, the declaratives on the top row have exactly one maximal supporting state: the set of all worlds where they are true.

At the same time, the clause also delivers conditional interrogatives like $p \to ?q$, which is resolved precisely when one of $p \to q$ and $p \to \neg q$ is established.

Now consider the case of an interrogative antecedent μ. If the consequent is a declarative α, the whole conditional is declarative, thus fully truth-conditional; hence, μ may simply be replaced by its presupposition: $\mu \to \alpha \equiv \pi_\mu \to \alpha$.

If the consequent is itself an interrogative ν, on the other hand, the clause says that $\mu \to \nu$ is resolved in s in case, if we extend s so as to resolve μ, the resulting state resolves ν. So, we can resolve $\mu \to \nu$ if we can resolve ν *conditionally* on having a resolution of μ, i.e., if our information is such that a resolution of μ determines a resolution of ν. E.g., the conditional interrogative $?p \to ?q$ is resolved precisely in case at least one of the following declaratives is established:

1. $(p \to q) \wedge (\neg p \to q) \quad \equiv q$
2. $(p \to q) \wedge (\neg p \to \neg q) \equiv q \leftrightarrow p$
3. $(p \to \neg q) \wedge (\neg p \to q) \quad \equiv q \leftrightarrow \neg p$
4. $(p \to \neg q) \wedge (\neg p \to \neg q) \equiv \neg q$

which correspond precisely to the four ways in which a resolution to $?p$ may determine a resolution to $?q$. Thus, such a conditional interrogative asks for enough information to establish a certain *interrogative dependency*.

3 Entailment and interrogative dependencies

Entailment in InqD_π is defined in the natural way, as preservation of support.

Definition 9 (Entailment).
$\Phi \models \psi \iff$ *for any model M and state s, if $M, s \models \Phi$ then $M, s \models \psi$.*

To see what this notion captures, consider first entailment towards a declarative. Fact 3 implies that, in this case, only truth-conditional content matters.

Fact 6 $\Phi \models \alpha \iff$ *for any M and any world w : if $M, w \models \Phi$ then $M, w \models \alpha$.*

When the conclusion is a declarative, then, interrogative assumptions μ may be replaced by their presuppositions π_μ, which share the same truth conditions. Thus, entailment towards declaratives is essentially a declarative business. Moreover, since truth-conditions are standard, our logic is a conservative extension of classical logic.

Fact 7 (Conservativity) *Let Γ, α be part of a standard propositional language. Then $\Gamma \models \alpha \iff \Gamma$ entails α in classical propositional logic.*

Now consider entailment towards an interrogative. Recall that to support a formula, or a set, is to support some resolution of it (facts 4 and 5). It is then easy to see that Φ entails ψ iff every resolution of Φ entails some resolution of ψ.

Fact 8 $\Phi \models \psi \iff$ *for all $\Gamma \in \mathcal{R}(\varphi)$ there is an $\alpha \in \mathcal{R}(\psi)$ s.t. $\Gamma \models \alpha$.*

Decomposing Φ into a set Γ of declaratives and a set Λ of interrogatives, and assuming ψ is an interrogative μ, this tells us that $\Gamma, \Lambda \models \mu$ holds iff any resolution of all interrogatives in Λ, together with Γ, entails some resolution of μ; that is, if *given Γ, any resolution of the interrogatives in Λ determines some resolution of μ.* To illustrate this, consider the following example of valid entailment:

$$p \leftrightarrow q \wedge r,\ ?q \wedge ?r \ \models \ ?p$$

Given the declarative $p \leftrightarrow q \wedge r$, any resolution of the conjunctive question $?q \wedge ?r$ determines a resolution of $?p$: for instance, the resolution $q \wedge r$ determines the resolution p, the resolution $q \wedge \neg r$ determines the resolution $\neg p$, and so on. Thus, an interrogative dependency is captured as a particular case of entailment in InqD_π, involving an interrogative conclusion and some interrogative assumptions.

Now that we know that interrogative dependencies are a case of entailment, it should no longer surprise us that they are internalized in the language as implications, such as $?p \to ?q$. As we have seen, the resolutions of such formulas embody precisely the different possible ways in which the consequent may be determined by the antecedent. Notice that the specific *way* in which an interrogative is determined by another may in turn be one of the variables on which the resolution of a certain interrogative conclusion depends. For instance, the following entailment captures the fact that a resolution of $?q$ is determined once a resolution to $?p$ is given and a specific dependency of $?q$ on $?p$ is established.

$$?p,\ ?p \to ?q \models ?q$$

How about the case in which we have an interrogative conclusion and *no* interrogative assumption? Since a set of declaratives Γ is the only resolution of itself, Fact 8 has the following corollary: Γ entails an interrogative μ iff it establishes some particular resolution of μ, i.e., in case it settles μ in a particular way.

Fact 9 *If Γ is a set of declaratives, $\Gamma \models \psi \iff \Gamma \models \alpha$ for some $\alpha \in \mathcal{R}(\psi)$.*

Summing up, then, inquisitive entailment brings under the same umbrella three crucial and seemingly independent notions of a logic of information and issues: *standard declarative entailment*, *answerhood*, and *interrogative dependency*.

4 Derivation system

A natural deduction system for InqD_π is described in the table below. In this system, the standard connectives—conjunction, implication, and falsum—are all assigned their standard inference rules. Thus, the core logical features of the connectives are preserved when these connectives are generalized to interrogatives.[4]

What does *not* generalize, by contrast, is the double negation axiom: for, double negation elimination is valid only for formulas whose semantics is truth-conditional, i.e., for which being supported amounts to being true at every world.

Fact 10 (Double negation characterizes truth-conditionality)
$\neg\neg\varphi \to \varphi$ *is valid iff for all* M, s*:* $M, s \models \varphi \iff (M, w \models \varphi$ *for all* $w \in s)$

Conjunction

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi \wedge \psi}{\varphi} \qquad \frac{\varphi \wedge \psi}{\psi}$$

Implication

$$\frac{\begin{matrix}[\varphi] \\ \vdots \\ \psi\end{matrix}}{\varphi \to \psi} \qquad \frac{\varphi \quad \varphi \to \psi}{\psi}$$

Interrogative

$$\frac{\alpha_i}{?\{\alpha_1, \dots, \alpha_n\}} \qquad \frac{\begin{matrix}[\alpha_1] \\ \vdots \\ \varphi\end{matrix} \quad \dots \quad \begin{matrix}[\alpha_n] \\ \vdots \\ \varphi\end{matrix} \quad ?\{\alpha_1, \dots, \alpha_n\}}{\varphi}$$

Falsum

$$\frac{\bot}{\varphi}$$

Kreisel-Putnam axiom
$(\alpha \to ?\{\beta_1, \dots, \beta_n\}) \to ?\{\alpha \to \beta_1, \dots, \alpha \to \beta_n\}$

Double negation
$\neg\neg\alpha \to \alpha$

The rules for the interrogative operator are simply the usual ones for a disjunction. This is hardly surprising, since the semantics of ? is disjunctive. The introduction rule says that if we have established some α_i, then we have resolved $?\{\alpha_1, \dots, \alpha_n\}$. The elimination rule says that if we can infer φ from the assumption that α_i is established for each i, then we can infer φ from the assumption that $?\{\alpha_1, \dots, \alpha_n\}$ is resolved.[5] Finally, the last component of the system is the

[4] We refer to the introduction and elimination rule for a connective $\circ$ as $(\circ i)$ and $(\circ e)$.
[5] The standard rules for negation and disjunction, which are derived connectives in our system, are admissible, with one caveat: a disjunction may only be eliminated towards a declarative. This restriction marks the difference between $\vee$ and ? and prevents unsound derivations such as $p \vee \neg p \vdash ?p$.

Kreisel-Putnam axiom, which distributes an implication over an interrogative consequent, provided the antecedent is a declarative.[6]

Definition 10.
We write $P : \Phi \vdash \psi$ if P is a proof with conclusion ψ whose set of undischarged assumptions is included in Φ. We write $\Phi \vdash \psi$ if some proof $P : \Phi \vdash \psi$ exists. We say φ and ψ are provably equivalent*, notation $\varphi \dashv\vdash \psi$, if $\varphi \vdash \psi$ and $\psi \vdash \varphi$.*

As usual, proving soundness is a tedious but straightforward matter.

Theorem 1 (Soundness). *If $\Phi \vdash \psi$ then $\Phi \models \psi$.*

Next, the normal form result of Corollary 1 is provable in the system. For the lengthy but straightforward inductive proof, the reader is referred to [3].

Lemma 1. *For any φ, $\varphi \dashv\vdash ?\mathcal{R}(\varphi)$.*

As a corollary, $(?i)$ ensures that a formula is derivable from any of its resolutions.

Corollary 2. *If $\alpha \in \mathcal{R}(\varphi)$, then $\alpha \vdash \varphi$.*

The following theorem, central to the completeness proof, says that derivability shares the property of entailment expressed by Fact 8: from Φ we can derive ψ iff from any specific resolution Γ of Φ we can derive some resolution α of ψ.

Theorem 2 (Resolution theorem).
$\Phi \vdash \psi \iff$ *for all $\Gamma \in \mathcal{R}(\Phi)$ there exists some $\alpha \in \mathcal{R}(\psi)$ s.t. $\Gamma \vdash \alpha$.*

The proof is given in the appendix. The inductive proof of the left-to-right direction of the theorem is constructive, providing an effective procedure that, given a proof $P : \Phi \vdash \psi$ and a resolution Γ of Φ, produces a new proof $Q : \Gamma \vdash \alpha$ of a specific resolution α of ψ. This procedure is illustrated in the next section.

Let us now turn to prove completeness. In [3], completeness was proved by reducing it to the completeness of classical propositional logic. Here, by contrast, we will provide an explicit canonical model construction. This strategy has the merit of generalizing to extensions of the language, such as *inquisitive epistemic logic* [6], for which a reduction strategy would not be viable [2].

The worlds in our canonical model will be *complete theories of declaratives*, i.e., sets Γ of declaratives which are (i) closed under deduction of declaratives; (ii) consistent; and (iii) complete: for any declarative α, either α or $\neg\alpha$ is in Γ.

Definition 11 (Canonical model).
The canonical model for InqD_π is the model $M^c = \langle \mathcal{W}^c, V^c \rangle$, where:

- *the elements of $\mathcal{W}^c$ are the complete theories of declaratives;*
- *$V^c : \mathcal{W}^c \to \wp(\mathcal{P})$ is the map $V^c(\Gamma) = \{p \in \mathcal{P} \mid p \in \Gamma\}$*

[6] This axiom is tightly related to the axiom $(\neg\varphi \to \psi \vee \chi) \to (\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)$ first investigated in the context of intermediate logics in [13], whence the name.

The bridge between derivability and semantics is given by the following *support lemma*, stating that support at a state S in the canonical model amounts to derivability from the intersection $\bigcap S$ of all the theories in S (where $\bigcap\emptyset$ is the set of all formulas). The inductive proof is given in the appendix.

Lemma 2 (Support lemma). *For any S and φ, $M^c, S \models \varphi \iff \bigcap S \vdash \varphi$.*

As shown in the appendix, the support lemma allows us to use the canonical model to give counterexamples to entailment, and thus to prove completeness.

Theorem 3 (Completeness theorem). *If $\Phi \models \psi$, then $\Phi \vdash \psi$.*

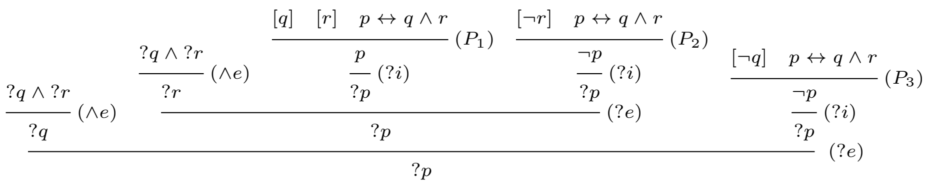
5 Computational content of inquisitive proofs

In this section we illustrate with an example the effective procedure described in the proof of the resolution theorem, and discuss the significance of this procedure. Consider again the example of valid entailment given in section 3:

$$p \leftrightarrow q \wedge r,\ ?q \wedge ?r \models ?p$$

Let $\Phi = \{p \leftrightarrow q \wedge r,\ ?q \wedge ?r\}$. By Fact 8, the validity of this entailment implies that any specific resolution Γ of Φ is bound to entail some resolution α of $?p$. The proof of the resolution theorem, then, tells us how to use a proof $P : \Phi \vdash ?p$ to produce a proof of some specific $\alpha \in \mathcal{R}(\psi)$ from a given $\Gamma \in \mathcal{R}(\Phi)$.

Below we have a proof $P : \Phi \vdash ?p$ in our deduction system. Subproofs involving only classical propositional logic have been omitted and denoted P_1, P_2, P_3.



Suppose now that we are given the specific resolution $\Gamma = \{p \leftrightarrow q \wedge r,\ q \wedge r\}$ of Φ, that is, suppose $?q \wedge ?r$ is resolved to $q \wedge r$, and let us see how we can use P to build a proof Q from Γ which yields either p or $\neg p$, whichever is entailed by Γ. The procedure builds the proof Q inductively on the structure of P, as follows:

1. replace assumptions of $?q \wedge ?r$ in P by the corresponding resolution $q \wedge r$;
2. where $(\wedge e)$ was used in P to get $?q$ and $?r$ from $?p \wedge ?r$, use the same rules to obtain q and r from $q \wedge r$; this gives us two proofs $Q_1 : \Gamma \vdash q$ and $Q_2 : \Gamma \vdash r$;
3. the proofs P_1, P_2 and P_3, which only involve declaratives, are left unchanged;
4. the step of ?-introduction right below these proofs is simply erased;
5. where in P the rule $(?e)$ is used to eliminate from $?r$, take the proof $Q_2 : \Gamma \vdash r$ obtained above and plug it in place of the assumption of r in $P_1 : \Gamma \cup \{q, r\} \vdash p$. In this way we obtain a proof $Q_3 : \Gamma \cup \{q\} \vdash p$;

6. similarly, where $(?e)$ was used to eliminate from $?q$, take the proof $Q_1 : \Gamma \vdash q$ that we have obtained and plug it in place of the assumption of q in Q_3.

The outcome is the following proof $Q : \Gamma \vdash p$, showing that Γ determines the resolution p of $?p$.

$$\dfrac{\dfrac{q \wedge r}{q}\,(\wedge e) \quad \dfrac{q \wedge r}{r}\,(\wedge e) \quad p \leftrightarrow q \wedge r}{p}\,(P_1)$$

Thus, the proof $P : \Phi \vdash ?p$ above does more than just witnessing the validity of the entailment: it encodes the *specific way* in which a resolution of $?p$ may be obtained, given $p \leftrightarrow q \wedge r$, from a resolution of $?q \wedge ?r$, that is, it provides a *method* for turning a resolution of $?q \wedge ?r$ into a corresponding resolution of $?p$.

Summing up, an inquisitive proof concluding with an interrogative μ builds up a way of resolving μ, dependent on resolutions to certain interrogative assumptions. Hence, while the declarative fragment of the logic is completely classical, when it comes to interrogatives, inquisitive proofs have a distinctive constructive content. This resonates with the observation that, in the realm of interrogatives, the double negation axiom is no longer valid. Indeed, our natural deduction system is essentially a system for intuitionistic logic—where ? plays the role of disjunction—with certain syntactic restrictions on formulas, and two extra ingredients: declarative double negation, and the Kreisel-Putnam axiom.[7]

6 Inquisitive logic as a logic of dependencies

We have seen how inquisitive logic captures interrogative dependencies as a case of entailment. In recent years, the logical study of dependency, sparked first by the study of bound variables in first-order logic, has received increasing attention, leading to the development of *dependence logic* [15, 7, 8, a.o.]. Lately, it has been realized [17] that propositional inquisitive semantics and propositional dependence logic are close relatives: they have the same semantic structures, identical expressive power, and differ only in their repertoire of logical operators.

Propositional dependence logic starts with dependence atoms $=(p_1, \dots, p_n, q)$, expressing the fact that the value of an atom q is determined by the values of $p_1, \dots, p_n$. This is a particular kind of interrogative dependency, which may be expressed in inquisitive semantics by means of the formula $?p_1 \wedge \dots \wedge ?p_n \rightarrow ?q$. So, in inquisitive semantics, the dependence *atom* actually has a *complex* logical structure, consisting of an implication, a determining antecedent—a conjunction

[7] The syntactic restrictions are only needed to keep the language dichotomous, and are not essential: we could replace the operator ? by a binary inquisitive disjunction $\vee$, and allow the connectives to apply unrestrictedly. We would then end up with the system InqB of *basic inquisitive semantics* [1, 5] , based on a simple propositional language. The results we have seen—in particular, the resolution theorem—translate, *mutatis mutandis*, to the setting of InqB. Thus, the properties described in this paper pertain to inquisitive logic at large, rather than just to the specific system InqD_π.

of atomic polar interrogatives—and a determined consequent—an atomic polar interrogative. I would like to argue here that, for a propositional logic of dependencies, the inquisitive repertoire of logical operations is a mathematically natural choice as well as a practically advantageous one.

First, once we recognize that dependencies are captured as a special case of entailment, it seems natural to internalize them in the language by means of an implication operation, especially since, perhaps surprisingly, such an operator exists and enjoys a natural semantics and a standard proof-theory.

One practical advantage of expressing dependencies by means of an implication, in combination with other operators, is that it becomes immediately clear that there is no need to restrict this operation to atomic polar interrogatives, or to polar interrogatives... or even to interrogatives. Implication makes sense in a much broader context, including the context of declaratives, in which we recover standard material implication. This extra generality allows us to express easily a wide spectrum of dependencies. For instance, the implication $?\{p, q\} \to ?\{r, s\}$ is supported in case every resolution to $?\{p, q\}$ determines a resolution to $?\{r, s\}$, that is, in case establishing either of p and q determines one of r and s.

Second, if we look at the set of *inquisitive propositions*—the non-empty, downward closed sets of states, which are the meaning objects of both inquisitive and propositional dependence logic—and we consider them ordered by entailment, we find that this space has a natural algebraic structure, namely, it forms a Heyting algebra [14]; the connectives of inquisitive semantics perform precisely the natural operations in this algebra: they express *meet*, *join*, and (algebraic) *implication*, which are responsible for the logical properties of these connectives.

As a concrete advantage, this principled treatment of connectives yields the well-behaved logical calculus described in this paper, and the one described in [1, 5] for the system InqB. Indeed, when it comes to proof theory, the dependence atoms are not ruled by simple logical laws: to axiomatize their logic, syntactically involved rules are needed, which essentially amount to an implicit stipulation of their resolutions [17]. By contrast, in inquisitive logic, formulas expressing dependencies are complex objects, built up by several logical operations—implication, conjunction, and question mark/inquisitive disjunction—each of which has natural, even standard inference rules. Taken together, these simple rules determine the complex logical properties manifested by dependence formulas. This lends further support to the view that the operations expressed by these connectives are the natural building blocks in the given semantic setting.

Conclusions I would like to close recapitulating the three main conclusions drawn in this paper: first, the relation of interrogative dependency is a particular case of entailment, involving interrogative conclusions and interrogative assumptions. Second, as a case of entailment, dependencies are internalized in the logical language as implications among interrogatives. Lastly, the associated logical calculus is a specific kind of constructive logic, whose proofs encode dependencies, and provide methods to compute a resolution of the interrogative conclusion from resolutions of the interrogative assumptions.

Appendix

Proof of theorem 2 (Resolution theorem)
Let us start from the left-to-right direction of the theorem: if Φ derives ψ, then any resolution Γ of Φ derives some resolution α of ψ.

The proof is a constructive one: given a proof $P : \Phi \vdash \psi$ and a resolution Γ of Φ, we show how to use P and Γ to build a proof $Q : \Gamma \vdash \alpha$ of some resolution α of ψ. The construction proceeds by induction on the structure of the proof P. We distinguish a number of cases depending on the last rule applied in P.

- ψ is an undischarged assumption, $\psi \in \Phi$. In this case, any resolution Γ of Φ contains a resolution α of ψ by definition, so $\Gamma \vdash \alpha$.
- ψ is an axiom. If ψ is an instance of the double negation axiom, then it is a declarative, and the claim is trivially true. If ψ is an instance of the Kreisel-Putnam axiom, of the form $(\beta \to ?\{\gamma_1, \dots, \gamma_n\}) \to ?\{\beta \to \gamma_1 \dots, \beta \to \gamma_n\}$, take $\alpha = \bigwedge_{1 \leq i \leq n}((\beta \to \gamma_i) \to (\beta \to \gamma_i))$: α is a resolution of ψ and, being a classical tautology, we have $\Gamma \vdash \alpha$ for any set Γ whatsoever, in particular for any $\Gamma \in \mathcal{R}(\Phi)$.
- $\psi = \chi \wedge \xi$ was obtained by $(\wedge i)$ from χ and ξ. Take any resolution Γ of Φ. Since the set of undischarged assumptions above both χ and ξ is included in Φ, by induction hypothesis from Γ we can deduce a resolution β of χ and a resolution γ of ξ. But then, by applying a conjunction introduction rule, from Γ we can deduce $\beta \wedge \gamma$, which is a resolution of ψ.
- $\psi = \chi \to \xi$ was obtained by $(\to i)$. Then the immediate subproof of P is a proof of ξ from the set of assumptions $\Phi \cup \{\chi\}$. Take any resolution Γ of Φ. Suppose $\alpha_1, \dots, \alpha_n$ are the resolutions of χ. For any $1 \leq i \leq n$, then, $\Gamma \cup \{\alpha_i\}$ is a resolution of $\Phi \cup \{\chi\}$, whence by induction hypothesis we have a proof $Q_i : \Gamma \cup \{\alpha_i\} \vdash \beta_i$ for some resolution β_i of ξ. But then, extending Q_i with an implication introduction, we derive $\alpha_i \to \beta_i$ from Γ. And since this is the case for any $1 \leq i \leq n$, from Γ we can derive $(\alpha_1 \to \beta_1) \wedge \dots \wedge (\alpha_n \to \beta_n)$, where $\{\beta_1, \dots, \beta_n\} \subseteq \mathcal{R}(\xi)$, which is a resolution of $\chi \to \xi = \psi$.
- $\psi = ?\{\alpha_1, \dots, \alpha_n\}$ was obtained by $(?i)$ from α_i. Thus, the immediate subproof of P is a proof of α_i from Φ. Take any resolution Γ of Φ. By induction hypothesis, from Γ we can then derive a resolution of α_i, that is to say, we can derive α_i, since a declarative is the only resolution of itself. So, the induction hypothesis gives us a proof of α_i from Γ, which is what we needed, since α_i is a resolution of ψ.
- ψ was obtained by $(\wedge e)$ from $\psi \wedge \chi$. Then the immediate subproof of P is a proof of $\psi \wedge \chi$ from Φ. Take a resolution Γ of Φ. By induction hypothesis there is a proof $Q : \Gamma \vdash \alpha$ for some resolution α of $\psi \wedge \chi$. By definition, such a resolution is of the form $\alpha = \beta \wedge \gamma$ where β is a resolution of ψ and γ is a resolution of χ. Extending Q with a rule of conjunction elimination, then, we have a proof of the resolution β of ψ from Γ. Of course, the argument is analogous if ψ was obtained by $(\wedge e)$ from a conjunction $\chi \wedge \psi$.
- ψ was obtained by $(\to e)$ from χ and $\chi \to \psi$. Then the immediate subproofs of P are a proof of χ from Φ, and a proof of $\chi \to \psi$ from Φ. Consider a

resolution Γ of Φ. By induction hypothesis we have a proof $Q_1 : \Gamma \vdash \beta$ where $\beta \in \mathcal{R}(\chi)$, and a proof $Q_2 : \Gamma \vdash \gamma$, where $\gamma \in \mathcal{R}(\chi \to \psi)$. Now, if $\mathcal{R}(\chi) = \{\beta_1, \ldots, \beta_n\}$, then $\beta = \beta_i$ for some i, and (by definition of the resolutions of an implication) $\gamma = (\beta_1 \to \gamma_1) \wedge \ldots \wedge (\beta_n \to \gamma_n)$ where $\{\gamma_1, \ldots, \gamma_n\} \subseteq \mathcal{R}(\psi)$. Now, extending Q_2 with an application of $(\wedge e)$ we get a proof of $\beta_i \to \gamma_i$ from Γ. Finally, putting together this proof with $Q_1 : \Gamma \vdash \beta_i$ and applying $(\to e)$, we obtain the resolution γ_i of $\mathcal{R}(\psi)$.

- ψ was obtained by $(?e)$ from a basic interrogative $?\{\beta_1, \ldots, \beta_m\}$. Then the immediate subproofs of P are a proof $P_0 : \Phi \vdash ?\{\beta_1, \ldots, \beta_m\}$ and, for $1 \leq i \leq n$ a proof $P_i : \Phi \cup \{\beta_i\} \vdash \psi$. Now consider a resolution Γ of Φ. By induction hypothesis we have a proof $Q_0 : \Gamma \vdash \beta$ for some $\beta \in \mathcal{R}(?\{\beta_1, \ldots, \beta_m\})$. Moreover, for any $1 \leq i \leq n$, since $\Gamma \cup \{\beta_i\}$ is a resolution of $\Phi \cup \{\beta_i\}$, by induction hypothesis we have a proof $Q_i : \Gamma \cup \{\beta_i\} \vdash \alpha_i$ where $\alpha_i \in \mathcal{R}(\psi)$. Now since β is a resolution of $?\{\beta_1, \ldots, \beta_m\}$, by definition $\beta = \beta_i$ for some i. But then, substituting any undischarged assumption of β_i in Q_i with an occurrence of the proof Q_0 with conclusion β_i, we obtain a proof of α_i from Γ, which is what we needed, since $\alpha_i \in \mathcal{R}(\psi)$.
- ψ was obtained by $(\bot e)$. This means that the immediate subproof of P is a proof of $\bot$ from Φ. Take any resolution Γ of Φ. By induction hypothesis, from Γ we can prove a resolution of $\bot$, that is, since a declarative is the only resolution of itself, we can prove $\bot$. But then, by an application of $(\bot e)$, from Γ we can obtain any formula, in particular any of the resolutions of ψ (crucially, the set of resolutions of a formula is always non-empty).

This completes the description of the inductive procedure to construct the required proof Q, and thus proves the left-to-right direction of the theorem. In order to establish the converse, let us make a detour through the following lemma.

Lemma 3. *If $\Phi \nvdash \psi$ then there exists a $\Gamma \in \mathcal{R}(\Phi)$ such that $\Gamma \nvdash \psi$.*

Proof. First assume Φ is finite. By induction on the number of formulas in Φ, we prove that *for any* ψ, if $\Phi \nvdash \psi$ there is some $\Gamma \in \mathcal{R}(\Phi)$ such that $\Gamma \nvdash \psi$.

If $\Phi = \emptyset$, the claim is trivially true. Now make the inductive hypothesis that the claim is true for sets of n formulas, and let us consider a set Φ of $n+1$ formulas. Then Φ is of the form $\Psi \cup \{\chi\}$ for some set Ψ of n formulas and some formula χ. Now consider a formula ψ such that $\Psi, \chi \nvdash \psi$. By Lemma 1, we must also have $\Psi, ?\mathcal{R}(\chi) \nvdash \psi$ whence, by the ?-introduction rule, we must have $\Psi, \alpha \nvdash \psi$ for some $\alpha \in \mathcal{R}(\chi)$. By the rules for implication, we must then have $\Psi \nvdash \alpha \to \psi$. So, by induction hypothesis there is a $\Gamma \in \mathcal{R}(\Psi)$ such that $\Gamma \nvdash \alpha \to \psi$. Finally, again by the rules for implication we have $\Gamma, \alpha \nvdash \psi$, which proves the claim since $\Gamma \cup \{\alpha\}$ is a resolution of $\Psi \cup \{\chi\}$.

Our inductive proof is thus complete, and the claim is proved for the case in which Φ is finite. The conclusion can then be extended to the infinite case by an argument using König's lemma. For the details, we refer to [2].

Proof of theorem 2, continued. What remained to be shown is the right-to-left direction of the theorem: if every resolution of Φ derives some resolution of ψ, then $\Phi \vdash \psi$. Contrapositively, suppose $\Phi \nvdash \psi$. By Lemma 3 there exists a $\Gamma \in \mathcal{R}(\Phi)$ such that $\Gamma \nvdash \psi$. Now, since for any $\alpha \in \mathcal{R}(\psi)$ we have $\alpha \vdash \psi$ (Corollary 2), Γ cannot derive any $\alpha \in \mathcal{R}(\psi)$, otherwise it would derive ψ as well. So, Γ is a resolution of Φ which does not derive any resolution of ψ. □

Lemma 4. *For any state $S \subseteq \mathcal{W}^c$ and declarative α, $\bigcap S \vdash \alpha \iff \alpha \in \bigcap S$*

Proof. If $\alpha \in \bigcap S$ then obviously $\bigcap S \vdash \alpha$. For the converse, suppose $\bigcap S \vdash \alpha$. For any $\Gamma \in S$ we have $\bigcap S \subseteq \Gamma$, so also $\Gamma \vdash \alpha$. But then, because Γ is closed under deduction of declaratives, we must have $\alpha \in \Gamma$. So, $\alpha \in \bigcap S$.

Proof of Lemma 2 (Support lemma)

The proof goes by induction on the complexity of φ. The straightforward cases for atoms, falsum, and conjunction are omitted.

Implication Suppose $\bigcap S \vdash \varphi \to \psi$. Take any $T \subseteq S$: if $T \models \varphi$ then by induction hypothesis $\bigcap T \vdash \varphi$. Since $T \subseteq S$, we have $\bigcap T \supseteq \bigcap S$, and since $\bigcap S \vdash \varphi \to \psi$, also $\bigcap T \vdash \varphi \to \psi$. By $(\to e)$, from $\bigcap T \vdash \varphi \to \psi$ and $\bigcap T \vdash \varphi$ it follows $\bigcap T \vdash \psi$, which by induction hypothesis implies $T \models \psi$. So, every substate of S that supports φ also supports ψ, that is, $S \models \varphi \to \psi$.
Viceversa, suppose $\bigcap S \nvdash \varphi \to \psi$. By the introduction rule for implication, this means that $\bigcap S, \varphi \nvdash \psi$. Now by Lemma 3 there is a a resolution of $(\bigcap S) \cup \{\varphi\}$ which does not derive ψ. Since $\bigcap S$ is a set of declaratives, this resolution must include a set of the form $(\bigcap S) \cup \{\alpha\}$ where α is a resolution of φ. Hence, there must exist a resolution α of φ such that $\bigcap S, \alpha \nvdash \psi$.
Now let $T = \{\Gamma \in S \,|\, \alpha \in \Gamma\}$. First, by definition we have $\alpha \in \bigcap T$, whence $\bigcap T \vdash \varphi$ by Corollary 2. By induction hypothesis we then have $T \models \varphi$. Now, if we can show that $\bigcap T \nvdash \psi$ we are done. For then, the induction hypothesis gives $T \not\models \psi$, which means that T is a substate of S that supports φ but not ψ, which shows that $S \not\models \varphi \to \psi$.
So, we are left to show that $\bigcap T \nvdash \psi$. Towards a contradiction, suppose that $\bigcap T \vdash \psi$. Since $\bigcap T$ is a set of declaratives, Theorem 2 tells us that $\bigcap T \vdash \beta$ for some resolution β of ψ, which by Lemma 4 amounts to $\beta \in \bigcap T$. So, for any $\Gamma \in T$ we have $\beta \in \Gamma$ and thus also $\alpha \to \beta \in \Gamma$, since Γ is closed under deduction of declaratives and $\beta \vdash \alpha \to \beta$. Now consider any $\Gamma \in S - T$: this means that $\alpha \notin \Gamma$; then since Γ is complete we have $\neg\alpha \in \Gamma$, whence $\alpha \to \beta \in \Gamma$, again because Γ is closed under deduction of declaratives and $\neg\alpha \vdash \alpha \to \beta$. This would mean, then, that $\alpha \to \beta \in \Gamma$ for any $\Gamma \in S$, whether $\Gamma \in T$ or $\Gamma \in S - T$. We can then conclude $\alpha \to \beta \in \bigcap S$, whence $\bigcap S, \alpha \vdash \beta$. And since β is a resolution of ψ we also have $\bigcap S, \alpha \vdash \psi$. But this is a contradiction, since by assumption α is such that $\bigcap S, \alpha \nvdash \psi$.

Interrogative operator If $S \models ?\{\alpha_1, \ldots, \alpha_n\}$, then $S \models \alpha_i$ for some i, so by induction hypothesis we have $\bigcap S \vdash \alpha_i$ and by ?-introduction also $\bigcap S \vdash ?\{\alpha_1, \ldots, \alpha_n\}$. Conversely, suppose $\bigcap S \vdash ?\{\alpha_1, \ldots, \alpha_n\}$. Since $\bigcap S$ is a set of declaratives, Theorem 2 implies $\bigcap S \vdash \alpha_i$ for some $1 \leq i \leq n$. By induction hypothesis we then have $S \models \alpha_i$, and thus also $S \models ?\{\alpha_1, \ldots, \alpha_n\}$. □

Proof of Theorem 3 (Completeness theorem)
Suppose $\Phi \nvdash \psi$. By Theorem 2, there is a resolution Θ of Φ which does not derive any resolution of ψ. Let $\mathcal{R}(\psi) = \{\alpha_1, \ldots, \alpha_n\}$: for each i, since $\Theta \nvdash \alpha_i$, the set $\Theta \cup \{\neg\alpha_i\}$ is consistent, and thus extendible to a complete theory of declaratives $\Gamma_i \in W^c$. Let $S = \{\Gamma_1, \ldots, \Gamma_n\}$ be the set of theories obtained for $1 \leq i \leq n$. It is easy to verify that, by construction, $S \models \Phi$ but $S \not\models \psi$, whence $\Phi \not\models \psi$. □

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