Dependency as Question Entailment

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Logics for Dependence and Independence — Dagstuhl, 23 June 2015

Scenario

- A disease may give rise to two symptoms, S_1 and S_2.
- S_2 much worse than S_1.
- The disease can be countered by means of a treatment T.
- However, T carries some associated risk.

A hospital's protocol

- Symptom S_2 $\rightarrow$ T always prescribed.
- Only symp. S_1 $\rightarrow$ T prescribed only if physical conditions are good.

Dependency

ν : whether the treatment is to be prescribed

is determined by

μ_1 : what symptoms the patient has

μ_2 : whether the patient is in good physical conditions

Dependency

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is determined by

μ_1 : what symptoms the patient has

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Observations

- μ_1, μ_2 and ν are questions.
- The dependency amounts to a relation between them: settling μ_1 and μ_2 implies settling ν.

Claim

Once classical logic is generalized to questions,
this relation emerges as a facet of entailment.

Part I

Dependency as question entailment

Traditional foundation for classical logic

- Traditionally, logic is concerned with statements.
- Sentence meaning = truth conditions.
- Semantics consists in a relation $w \models \alpha$ of truth between possible worlds w and statements α.
- Entailment amounts to preservation of truth:

$$\Gamma \models \alpha \iff \text{for all } w: \; w \models \Gamma \text{ implies } w \models \alpha$$

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- Question meaning is not truth-conditional.
  So, apparently no role for questions in logic.

An informational foundation for classical logic

- We can give an alternative, informational foundation for classical logic.
- Take the meaning of α to be given by laying out what information it takes to settle α.
- We will model information states as sets of worlds.
- Semantics consists in a relation $s \models \alpha$ called support between info states and sentences.
- Entailment amounts to preservation of support:

$$\Gamma \models \alpha \iff \text{for all } s: \; s \models \Gamma \text{ implies } s \models \alpha$$

An informational foundation for classical logic

- Given our modeling of information, truth and support are interdefinable:
  - $s \models \alpha \iff w \models \alpha$ for all $w \in s$
  - $w \models \alpha \iff \{w\} \models \alpha$

- As a consequence, truth-entailment and support-entailment coincide.

- Support semantics is an alternative foundation for classical logic.

Support semantics makes it possible to interpret questions

- Unlike truth-conditional semantics, support semantics interprets questions.
- It makes sense to ask whether a question is settled in an information state.

Whether the patient is in good conditions

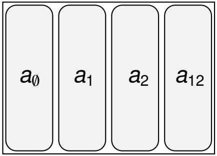
$s \models \mu_2 \iff s \subseteq a_g$ or $s \subseteq \overline{a_g}$

- $a_g = \{w \mid \text{patient in good conditions in } w\}$
- $\overline{a_g} = \{w \mid \text{patient not in good conditions in } w\}$

What symptoms the patient has

$s \models \mu_1 \iff s \subseteq a_i$ for some a_i, where:

- $a_\emptyset = \{w \mid \text{patient has no symptoms in } w\}$
- $a_1 = \{w \mid \text{patient has only } S_1 \text{ in } w\}$
- $a_2 = \{w \mid \text{patient has only } S_2 \text{ in } w\}$
- $a_{12} = \{w \mid \text{patient has both symptoms in } w\}$



Entailment

- $\Phi \models \psi \iff \forall s: \; s \models \Phi \text{ implies } s \models \psi$
- Since support conditions are defined also for questions, questions can now take part in entailments.

Entailment among statements

$\alpha \models \beta \iff$ settling that α implies settling that β

$\iff \beta$ is a truth-conditional consequence of α

Entailment among questions

$\mu \models \nu \iff$ settling μ implies settling ν

$\iff \mu$ logically determines ν

Entailment in context

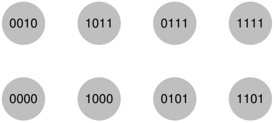
- We model a context as an information state s.
- Entailment relative to s only takes into account worlds $w \in s$:

$$\Phi \models_s \psi \iff \forall t \subseteq s : \; t \models \Phi \text{ implies } t \models \psi$$

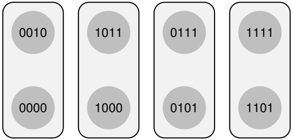
- Contextual question entailment captures dependencies that hold on the basis of specific assumptions.

$$\mu \models_s \nu \iff \mu \text{ determines } \nu \text{ in context } s$$

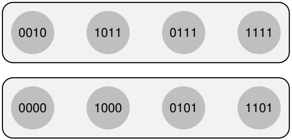
Back to our example



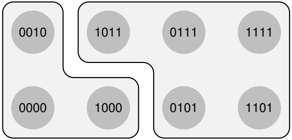
s = protocol context



μ_1 = symptoms?



μ_2 = conditions?



ν = treatment?

$\mu_1 \not\models_s \nu$ $\quad$ $\mu_2 \not\models_s \nu$ $\quad$ $\mu_1, \mu_2 \models_s \nu$

Internalizing entailment

- In support semantics, the contexts for entailment are the same objects at which formulas are evaluated.

- A support-based logic can always be equipped with an operation of implication which internalizes entailment:

$$s \models \varphi \to \psi \iff \varphi \models_s \psi$$

- Spelling out the definition of $\varphi \models_s \psi$, we get the inductive support clause for $\varphi \to \psi$:

$$s \models \varphi \to \psi \iff \forall t \subseteq s : \; t \models \varphi \text{ implies } t \models \psi$$

Internalizing entailment

- On statements, $\to$ is just the familiar material conditional:

$$s \models \alpha \to \beta \iff \text{for all } w \in s: \; w \not\models \alpha \text{ or } w \models \beta$$

- On the other hand, now $\to$ also makes sense on questions:

$$s \models \mu \to \nu \iff \mu \models_s \nu \iff \mu \text{ determines } \nu \text{ in } s$$

- Just like entailment generalizes to questions to capture dependency, $\to$ generalizes to questions, allowing us to express dependencies.

Taking stocks

- The move from truth-conditions to support-conditions brings questions within the purview of logic.
- Entailment becomes meaningful for questions as well.
- Dependency emerges as question entailment in context.
- Since entailment is internalized by an implication operation, dependencies are expressible as implications between questions.

Part II

Propositional logic

Questions in propositional logic

- We will now see how the ideas of support semantics are implemented in a concrete formal system.
- This is essentially the system InqB of propositional inquisitive logic.
- We will look at InqB from a new perspective:
  - usually, InqB is regarded as a non-classical logic, in which propositional formulas have a more fine-grained meaning;
  - here, we will look at InqB as a conservative extension of classical propositional logic with questions.

Worlds and states for propositional logic

- Let $\mathcal{P}$ denote a set of propositional letters.
- A possible world is a propositional valuation $w : \mathcal{P} \to \{0, 1\}$.
- An information state is a set of propositional valuations.

Classical formulas

$\mathcal{L}_{cl} \quad ::= \quad p \quad | \quad \bot \quad | \quad \varphi \wedge \varphi \quad | \quad \varphi \to \varphi$

Defined connectives

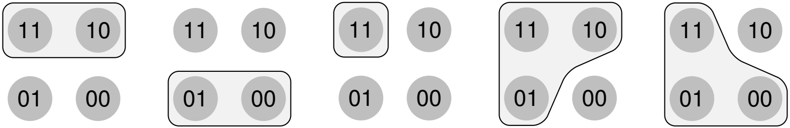
- $\neg\varphi := \varphi \to \bot$
- $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$

Support

- $s \models p \iff \forall w \in s : \; w(p) = 1$
- $s \models \bot \iff s = \emptyset$
- $s \models \varphi \wedge \psi \iff s \models \varphi$ and $s \models \psi$
- $s \models \varphi \to \psi \iff \forall t \subseteq s : \; t \models \varphi$ implies $t \models \psi$

Classical formulas are truth-conditional

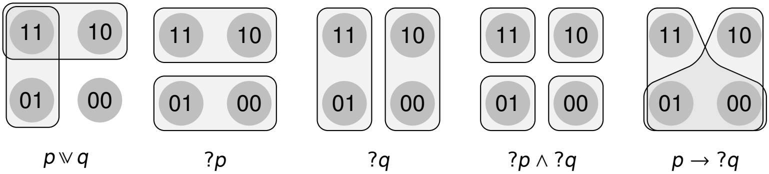
$s \models \alpha \iff \forall w \in s : w \models \alpha$



p $\quad$ $\neg p$ $\quad$ $p \wedge q$ $\quad$ $p \vee q$ $\quad$ $p \to q$

Extending CPL with questions

- We expand our language with an inquisitive disjunction $\mathbin{\mathbb{V}}$.
- $s \models \varphi \mathbin{\mathbb{V}} \psi \iff s \models \varphi$ or $s \models \psi$
- $\alpha \mathbin{\mathbb{V}} \beta$ can be read as the question whether α or β.
- $?\alpha := \alpha \mathbin{\mathbb{V}} \neg\alpha$ can be read as the question whether α.



Expressing dependencies in InqB

- In propositional dependence logic, dependencies are expressed by special atoms $=(p_1, \dots, p_n, q)$.

$$
\begin{aligned}
s \models =(p, q) \quad &\iff \quad \forall w, w' \in s, \quad w(p) = w'(p) \text{ implies } w(q) = w'(q) \\
&\iff \quad \forall t \subseteq s, \quad t \models ?p \text{ implies } t \models ?q \\
&\iff \quad ?p \models_s ?q \\
&\iff \quad s \models ?p \to ?q
\end{aligned}
$$

- $=(p_1, \dots, p_n, q)$ captures entailment between atomic polar questions.
- $=(p_1, \dots, p_n, q)$ is broken down into basic operations: $\to$, $\wedge$, $\Vvee$, $\bot$
- These are the Heyting algebra operations on the underlying space of meanings. (cf. also Abramsky and Väänanen '09,Roelofsen '13)

- Dependence atoms are a particular case of a general pattern. For *any* questions μ, ν, $\mu \to \nu$ expresses that μ determines ν.

- Whether J will go to the party depends on whether one of B and S will.

  $$?(b \vee s) \;\to\; ?j$$

- Whether J or B is in the office depends on whether it is Sat or Sun.

  $$(sat \mathbin{\vee\!\!\!\vee} sun) \;\to\; (j \mathbin{\vee\!\!\!\vee} b)$$

- Whether M will dance if J asks her depends on whether she's in a good mood.

  $$?g \;\to\; (a \to ?d)$$

Part III

Reasoning with questions

Proof system

Conjunction

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi \wedge \psi}{\varphi} \quad \frac{\varphi \wedge \psi}{\psi}$$

Implication

$$\frac{\begin{array}{c}[\varphi] \\ \vdots \\ \psi\end{array}}{\varphi \to \psi} \qquad \frac{\varphi \quad \varphi \to \psi}{\psi}$$

Inquisitive disjunction

$$\frac{\varphi}{\varphi \mathbin{\mathbb{v}} \psi} \quad \frac{\psi}{\varphi \mathbin{\mathbb{v}} \psi} \qquad \frac{\varphi \mathbin{\mathbb{v}} \psi \quad \begin{array}{c}[\varphi] \\ \vdots \\ \chi\end{array} \quad \begin{array}{c}[\psi] \\ \vdots \\ \chi\end{array}}{\chi}$$

Falsum

$$\frac{\bot}{\varphi}$$

Kreisel-Putnam

$$\frac{\alpha \to \psi \mathbin{\mathbb{v}} \chi}{(\alpha \to \psi) \mathbin{\mathbb{v}} (\alpha \to \chi)} \; (\alpha \in \mathcal{L}_{cl})$$

Classical ¬¬ elimination

$$\frac{\neg\neg\alpha}{\alpha} \; (\alpha \in \mathcal{L}_{cl})$$

Back to our example

- Assume the following atomic sentences:

  s_1 : patient has symptom S_1

  s_2 : patient has symptom S_2

  g : patient is in good physical conditions

  t : treatment prescribed

- Our protocol is described by: $\gamma := t \leftrightarrow s_2 \vee (s_1 \wedge g)$

- Our dependency is captured by: $\gamma,\ ?s_1,\ ?s_2,\ ?g \models ?t$

A formal proof of our dependency

$$
\dfrac{
?s_2 \qquad
\dfrac{\dfrac{\gamma \quad [s_2]}{t}\,(\text{C1})}{?t}\,(\mathbin{\vee\!\!\!\vee}\text{i})
\qquad
\dfrac{
?s_1 \qquad
\dfrac{\dfrac{\gamma \quad [\neg s_1] \quad [\neg s_2]}{\neg t}\,(\text{C2})}{?t}\,(\mathbin{\vee\!\!\!\vee}\text{i})
\qquad
\dfrac{
?g \qquad
\dfrac{\dfrac{\gamma \quad [s_1] \quad [g]}{t}\,(\text{C3})}{?t}\,(\mathbin{\vee\!\!\!\vee}\text{i})
\qquad
\dfrac{\dfrac{\gamma \quad [\neg s_2] \quad [\neg g]}{\neg t}\,(\text{C4})}{?t}\,(\mathbin{\vee\!\!\!\vee}\text{i})
}{?t}\,(\mathbin{\vee\!\!\!\vee}\text{e})
}{?t}\,(\mathbin{\vee\!\!\!\vee}\text{e})
}{?t}\,(\mathbin{\vee\!\!\!\vee}\text{e})
$$

A formal proof of our dependency

$$\dfrac{?s_2 \qquad \dfrac{\dfrac{\gamma \quad [s_2]}{t}\,(\mathrm{C1})}{?t}\,(\mathbb{V}\mathrm{i}) \qquad \dfrac{?s_1 \qquad \dfrac{\dfrac{\gamma \quad [\neg s_1] \quad [\neg s_2]}{\neg t}\,(\mathrm{C2})}{?t}\,(\mathbb{V}\mathrm{i}) \qquad \dfrac{?g \qquad \dfrac{\dfrac{\gamma \quad [s_1] \quad [g]}{t}\,(\mathrm{C3})}{?t}\,(\mathbb{V}\mathrm{i}) \qquad \dfrac{\dfrac{\gamma \quad [\neg s_2] \quad [\neg g]}{\neg t}\,(\mathrm{C4})}{?t}\,(\mathbb{V}\mathrm{i})}{?t}\,(\mathbb{V}\mathrm{e})}{?t}\,(\mathbb{V}\mathrm{e})}{?t}\,(\mathbb{V}\mathrm{e})$$

- The proof shows that, given γ, the questions $?s_1$, $?s_2$ and $?g$ determine the question $?t$.
- It does so by actually describing how to turn resolutions to the assumptions into a corresponding resolution of the conclusion.

Resolutions

- $\mathcal{R}(p) = \{p\}$
- $\mathcal{R}(\bot) = \{\bot\}$
- $\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta \,|\, \alpha \in \mathcal{R}(\varphi)\}$
- $\mathcal{R}(\varphi \to \psi) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)} (\alpha \to f(\alpha)) \,|\, f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$
- $\mathcal{R}(\varphi \vee\!\!\!\vee \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$

Examples

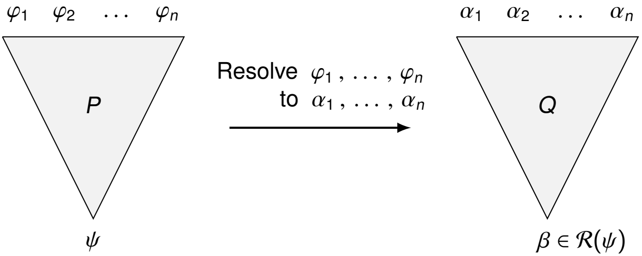
$\mathcal{R}(\alpha) = \{\alpha\}$ if $\alpha \in \mathcal{L}_c$ $\quad$ $\mathcal{R}(p \vee\!\!\!\vee q) = \{p, q\}$ $\quad$ $\mathcal{R}(?p) = \{p, \neg p\}$

Normal form

$\varphi \equiv \alpha_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \alpha_n$ where $\{\alpha_1, \ldots, \alpha_n\} = \mathcal{R}(\varphi)$

Resolution algorithm

Let $P : \overline{\varphi} \vdash \psi$ and $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$. There is a procedure which, inductively on P, constructs a proof $Q : \overline{\alpha} \vdash \beta$ having a resolution β of ψ as conclusion.



Example

Suppose the data for a certain patient are s_1, $\neg s_2$, g.
Applying the algorithm to our proof gives:

$$\frac{\gamma \quad s_1 \quad g}{t}\ (\text{C3})$$

Conclusion

Just like entailments involving questions capture dependencies, proofs involving questions encode algorithms for dependencies.

Part IV

First-order logic

Multi-model setting (standard inquisitive FO logic)

$s \models_g \varphi$, where s is a set of FO models, g an assignment.

Language

- Classical language: $\bot$, $\wedge$, $\to$, $\forall$
- Full language: $\bot$, $\wedge$, $\to$, $\forall$, $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, $\overline{\exists}$

Questions

$\overline{\exists} xPx$: what is an instance of a P?

$\forall x?Px$: what is the extension of P?

Dependencies

$\overline{\exists} xPx \to \overline{\exists} xQx$: any instance of P determines an instance of Q.

$\forall x?Px \to \forall x?Qx$: the extension of P determines the extension of Q.

Multi-assignment setting (standard dependence logic)

$M \models_X \varphi$, where M is a FO model, X a set of assignments.

Questions

$$\lambda_x := \overline{\exists} y(y = x)$$

λ_x : what is the value of x? (dependence logic's $=(x)$)

$?\mathrm{even}(x)$: is x even?

Dependencies

$\lambda_x \to \lambda_y$: the value of x determines the value of y. (captures the dependence atom $=(x, y)$).

$\lambda_x \to ?\mathrm{even}(y)$: the value of x determines the parity of y.

$?\mathrm{even}(x) \to \lambda_y$: the parity of x determines the value of y.

Multi-model and multi-assignment setting

$s \models \varphi$, where s is a set of model-assignment pairs.

Questions

- Questions about the model: $?Pa$
- Questions about the assignment: λ_x
- Hybrid questions: $?Px$

Dependencies

- Dependencies among features of the model: $?Pa \rightarrow ?Qa$
- Dependencies among features of the assignment: $\lambda_x \rightarrow \lambda_y$
- Mixed dependencies: $?Pa \rightarrow \lambda_y$

Wrapping up

- By taking an informational perspective on semantics, we can extend classical logic to encompass questions.
- Dependency then emerges as a facet of entailment: entailment among questions in context.
- This connection has a proof-theoretic side to it: proof involving questions encode methods for computing dependencies.

Wrapping up

- Entailment may be internalized by means of an implication operation:

$$\mu \models_s \nu \iff s \models \mu \to \nu$$

- Dependencies generally expressible as implications among questions.

- Dependence atoms are captured as particular cases:

$$=\!(p, q) \rightsquigarrow ?p \to ?q \qquad =\!(x, y) \rightsquigarrow \lambda_x \to \lambda_y$$

- But many other cases of dependencies may be expressed as well.
  - Whether p or q determines whether r or s: $p \mathbb{v} q \to r \mathbb{v} s$
  - The extension of P determines the extension of Q: $\forall x?Px \to \forall x?Qx$
  - The value of x determines the parity of y: $\lambda_x \to ?\text{even}(y)$


THANK
YOU
GRACIAS
ARIGATO
SHUKURIA
DANKSCHEEN
BİYAN
SHUKRIA
TINGKI
YAQHANYELAY
TASHAKKUR ATU
SUKSAMA
EKHMET
GRAZIE
MEHRBANI
PALDIES
MAAKE
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