Dependence statements are strict conditionals

Ivano Ciardelli

Munich Center for Mathematical Philosophy,
Ludwig-Maximilians-Universität, Munich

Abstract

In this paper I discuss dependence statements like "*whether p determines whether q*". I propose to analyze such statements as involving a generalized strict conditional operator applied to two questions—a determining question and a determined one. The dependence statement is true or false at a world w according to whether, relative to the set of successors of w, every answer to the former yields an answer to the latter. This motivates an investigation of strict conditionals in the context of inquisitive logic. A sound and complete axiomatization of the resulting logic is established, both for the class of all Kripke frames, and for various notable frame classes.

Keywords: Dependency, inquisitive logic, strict conditional, questions

1 Introduction

Consider a process which may have different outcomes, leading to different truth values for the sentences p, q, r. I will be concerned with dependence statements like (1-a) and sentences that embed such statements, like (1-b-e). [1]

(1) a. Whether q is determined by whether p.
b. Whether q is not determined by whether p.
c. Whether q is determined by whether p, or it is determined by whether r.
d. Alice knows that whether q is determined by whether p.
e. Is it the case that whether q is determined by whether p?

The main question I will address is how statements like (1-a) are best formalized in a logical language so that we get a satisfactory analysis of these statements as well as of compound sentences like (1-b-e) in which these statements are embedded under connectives, modalities, and a polar question operator.

The kind of dependency that is involved in these statements is an interesting logical notion, which plays an important role in science. In recent years, this notion has come under attention in logic from the perspective of two historically

[1] In agreement with the literature (e.g., [10]), I take the verb *determine* in (1) to mean *completely determine*, and to include the limit case in which the dependency is trivial (e.g., q is bound to be true regardless of p).

independent but formally related traditions. The first tradition is associated with *dependence logic*. Originating with Väänänen's work on dependencies between bound variables in predicate logic [18], dependence logic evolved into a general theory of logical systems equipped with the tools to reason about dependency relations. In particular, the system of propositional dependence logic [21] and its extension with modal operators [19,9] are equipped with formulas expressing dependencies between truth-values of sentences. E.g., the formula $=(p, q)$ expresses the fact that the truth-value of q is functionally determined by the truth-value of p. In order to interpret such formulas, dependence logic uses a non-standard semantics in which formulas are interpreted not with respect to single possible worlds, but rather with respect to sets of worlds, called *teams*.

The second tradition is associated with *inquisitive logic*. Originating from a line of work that aimed at a uniform analysis of statements and questions [11,12,14,7,5], inquisitive logic evolved into a full-blown theory of logic in a context encompassing both kinds of sentences (for an overview, see [3]). In this tradition, too, formulas are interpreted not with respect to single possible worlds, representing states of affairs, but with respect to sets of worlds, which are viewed as states of information. The idea is that the meaning of a question is given not by specifying in what states of affairs it is true, but rather by specifying what information is needed to resolve it. As for statements, their meaning can be given by specifying what information is needed to establish them. Thus, in an information-based semantics statements and questions can be interpreted in terms of a single semantic relation, called *support*. A fundamental connection between questions and dependency was established in [2,4]: dependency is nothing but question entailment in context. E.g., the dependency expressed by (1-a) amounts to the fact that, relative to the set of possible outcomes, the question whether p entails the question whether q. Due to a systematic connection between contextual entailment and the implication connective, this entailment can be expressed within the language by means of the formula $?p \to ?q$, which is equivalent to the dependence logic formula $=(p, q)$. Various advantages of this modular analysis of dependencies are discussed in [2] and in Ch. 5 of [3].

Nevertheless, in this paper I argue that the formula $?p \to ?q$ (and, thus, also the equivalent dependence logic formula $=(p, q)$) is not the right way to formalize a dependence statement like (1-a). Rather, I propose that dependence statements are modal statements. The appropriate semantic structures to interpret such statements are Kripke models, where each possible world w is equipped with a corresponding set $R[w]$ of possibilities. A dependence statement is essentially a strict conditional, which is true or false at a world w depending on some global features of the set $R[w]$. It is, however, a strict conditional whose antecedent and consequent are questions. Thus, e.g., (1-a) should be formalized as the formula $?p \Rightarrow ?q$, which is true at w just in case the question $?p$ entails the question $?q$ relative to $R[w]$. Making this idea work requires extending the strict conditional operator to a logic equipped with questions. In this paper I define and investigate such an extension.

The paper is structured as follows: Section 2 provides some background on propositional inquisitive logic, InqB; Section 3 describes a number of problems that arise if we analyze dependence statements as InqB-implications; Section 4 shows that these problems disappear if dependence statements are analyzed as strict conditionals involving questions; Section 5 investigates the logic of strict implication in the inquisitive setting, leading up to a sound and complete axiomatization; Section 6 provides a comparison with a recent related proposal.

2 Propositional inquisitive logic

Propositional inquisitive logic, InqB, can be seen as a conservative extension of classical propositional logic with questions. The language $\mathcal{L}$ of the system is given by the following BNF, where p ranges over a set $\mathcal{P}$ of atomic sentences:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi$$

Formulas which do not contain occurrences of $\mathbin{\vee\!\!\!\vee}$ are called *classical formulas.* The set $\mathcal{L}_c$ of classical formulas can be identified with the standard language of propositional logic. Negation and classical disjunction are defined by setting:

$$\neg\varphi := \varphi \to \bot \qquad \varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$$

The connective $\mathbin{\vee\!\!\!\vee}$, called *inquisitive disjunction*, allows us to form questions. In particular, the formula $?p := p \mathbin{\vee\!\!\!\vee} \neg p$ can be seen as a representation of the polar question *whether* p. In general, we set:

$$?\varphi := \varphi \mathbin{\vee\!\!\!\vee} \neg\varphi$$

A model for InqB is a pair $M = \langle W, V\rangle$ consisting of a set W of possible worlds and a valuation function $V : W \times \mathcal{P} \to \{0,1\}$. In order to interpret statements and questions in a uniform logical framework, InqB interprets formulas not in terms of truth relative to a possible world, but in terms of *support* relative to a set of worlds. Following a tradition that goes back to the work of Hintikka [13], such a set is referred to as an *information state*: the idea is that a set $s \subseteq W$ encodes a body of information which is compatible with the actual world being one of the worlds $w \in s$, and incompatible with it being one of the worlds $w \notin s$. The support relation for InqB is defined recursively as follows.

Definition 2.1 [Support for InqB]

- $M, s \models p$ iff $\forall w \in s : V(w, p) = 1$
- $M, s \models \bot$ iff $s = \emptyset$
- $M, s \models \varphi \wedge \psi$ iff $M, s \models \varphi$ and $M, s \models \psi$
- $M, s \models \varphi \mathbin{\vee\!\!\!\vee} \psi$ iff $M, s \models \varphi$ or $M, s \models \psi$
- $M, s \models \varphi \to \psi$ iff $\forall t \subseteq s : M, t \models \varphi$ implies $M, t \models \psi$

The support relation has the following basic properties: the first says that support is preserved as information increases; the second says that every formula is supported by the inconsistent information state, $s = \emptyset$.

- Persistency: if $M, s \models \varphi$ and $t \subseteq s$, then $M, t \models \varphi$;
- Semantic *ex-falso*: $M, \emptyset \models \varphi$ for all $\varphi \in \mathcal{L}$.

The derived clauses for negation and classical disjunction can be given naturally in terms of the relation $s \between \varphi$ of *compatibility* between a state s and a formula φ, which holds in case s can be extended consistently to a state t that supports φ:

- $s \between \varphi$ iff there exists a non-empty $t \subseteq s$ such that $t \models \varphi$
- $s \models \neg\varphi$ iff it is not the case that $s \between \varphi$
- $s \models \varphi \vee \psi$ iff for all non-empty $t \subseteq s$, $t \between \varphi$ or $t \between \psi$

Truth at a world w is recovered as a special case of support, i.e., support at $\{w\}$.

Definition 2.2 [Truth]
We say that φ is true at world w in M, notation $M, w \models \varphi$, if $M, \{w\} \models \varphi$. The truth-set of φ in M is the set $|\varphi|_M = \{w \in M \mid M, w \models \varphi\}$.

For some formulas, support at a state s amounts to truth at each world $w \in s$. Formulas with this property are said to be *truth-conditional* (or *flat*, in the dependence logic literature). They can be viewed as corresponding to statements, whose semantics is completely determined by their truth-conditions.

Definition 2.3 [Truth-conditionality]
We say that φ is *truth-conditional* if for every model M and state s:
$M, s \models \varphi$ iff $\forall w \in s : M, w \models \varphi$

It is easy to verify by induction that all classical formulas are truth-conditional. Moreover, the truth-conditions that Definition 2.2 assigns to them are just the ones familiar from classical logic. Thus, for classical formulas, the support semantics given here is essentially equivalent to the standard truth-conditional semantics. In this sense, when restricted to statements, the above definition just provides an alternative semantics for classical propositional logic. The benefit of this alternative semantic setup is that it allows us to interpret not only statements, but also *questions*, which we take to be formulas of $\mathcal{L}$ which are not truth-conditional (on the relation between questions and truth-conditions, see Ch. 1 of [3]). For an example, take the formula $?p := p \mathbin{\vee\!\!\!\vee} \neg p$. We have:

$$M, s \models ?p \text{ iff } s \subseteq |p|_M \text{ or } s \subseteq |\neg p|_M$$

That is, $?p$ is supported at s just in case the information available in s settles whether or not p is true, i.e., resolves the question whether p. In general, if μ is a question, the relation $s \models \mu$ captures the fact that s contains enough information to resolve μ.

Now let us come back to the relation of dependency discussed in the beginning of the paper. Let s be an information state. It is natural to say that a question λ *determines* a question μ relative to s if in context of s, resolving λ implies resolving μ; that is, if any way of extending s to a stronger state $t \subseteq s$ that supports λ leads to a state that also supports μ. Generalizing to multiple determining questions, we say that a set of questions Λ determines a question

μ in s if extending s to a state $t \subseteq s$ that supports all questions in Λ leads to a state that also supports μ. This relation of dependency can be seen as the special case of a more general relation of contextual entailment (see [3,4]).

Definition 2.4 [Contextual entailment]
A set of formulas Φ entails a formula ψ relative to an information state s, notation $\Phi \models_s \psi$, if for all states $t \subseteq s$, if $t \models \varphi$ for all $\varphi \in \Phi$, then $t \models \psi$.

The following remark shows that, if Φ is a finite set $\{\varphi_1, \ldots, \varphi_n\}$, the fact that $\varphi_1, \ldots, \psi_n$ contextually entail ψ can be expressed by an implication.

Remark 2.5 $s \models \varphi_1 \wedge \cdots \wedge \varphi_n \to \psi$ iff $\varphi_1, \ldots, \varphi_n \models_s \psi$

In particular, since dependencies are special cases of contextual entailment, they are expressible within the language as implications between questions. To make this more concrete, let us look at one specific example.

Example 2.6 Suppose we want to express that, in the state s, whether q is determined by whether p. This means that in s, resolving the question $?p$ implies resolving the question $?q$. The relevant property of s can then be expressed by the implication $?p \to ?q$. Indeed, we have:

$$M, s \models ?p \to ?q \;\text{ iff }\; \exists f : \{0,1\} \to \{0,1\} \text{ s.t. } \forall w \in s : V(w, q) = f(V(w, p))$$

In words, s supports $?p \to ?q$ if and only if within s the truth-value of q is functionally determined by the truth-value of p.

This example generalizes straightforwardly. For any classical formula α, let us denote by $V(w, \alpha)$ the truth-value of α at w (1 if $w \in |\alpha|_M$ and 0 otherwise). Let $\alpha_1, \ldots, \alpha_n, \beta$ be classical formulas: $?\alpha_1 \wedge \cdots \wedge ?\alpha_n \to ?\beta$ expresses that, in s, whether β is true is determined by which of the α_is are true:

$$M, s \models ?\alpha_1 \wedge \cdots \wedge ?\alpha_n \to ?\beta \;\text{ iff }\; \exists f : \{0,1\}^n \to \{0,1\} \text{ s.t. } \forall w \in s : V(w, \beta) = f(V(w, \alpha_1), \ldots, V(w, \alpha_n))$$

3 Dependence statements aren't inquisitive conditionals

The discussion in the previous section illustrates how InqB provides a simple and elegant language to express dependencies. This might suggest that we can analyze dependence statements in InqB, translating, e.g., (1-a) as $?p \to ?q$. However, this analysis of (1-a) leads to a number of problems.

Consider first the truth-conditions of this sentence. Intuitively, (1-a) may well be false at a given possible world. E.g., it is intuitively false at the actual world, if we are talking about the rolling of a die, and p and q are taken to be, respectively, "the outcome is even" and "the outcome is prime". By contrast, $?p \to ?q$ is true at every world in every model: this is because relative to a singleton $\{w\}$, the truth-value of q is functionally determined by that of p in a trivial way. Thus, the truth-conditions of $?p \to ?q$ do not match those of (1-a).

Next, consider the interaction of dependence sentences with other operators. First take an embedding under negation. InqB has a negation operator $\neg$, which is used to formalize negative statements like *Alice is not home*. If we translate (1-a) as $?p \to ?q$, we'd want to translate (1-b) as $\neg(?p \to ?q)$. But this does not work: while (1-b) is a consistent statement, $\neg(?p \to ?q)$ is a contradiction.

One may try to solve this problem by changing the clause for negation, letting $s \models \neg p$ iff $s \not\models p$. However, this would not work as a general account of negation: in order to establish that *Alice is not home*, it is not enough that we have not established that she is, as this clause would have it; we must really exclude the possibility that she is home, as required by the InqB-negation. Thus, we would need two different negations, one to translate negations of ordinary statements, and the other to translate negations of dependence statements. This seems undesirable. While an extension of InqB with both negations has indeed been studied and axiomatized [15], the solution proposed here shows that it is possible to use a single negation in all cases. There is no need to stipulate a special meaning for negation when it applies to dependence statements.

Second, consider the case of a disjunctive dependence statement like (1-c). If we translate (1-a) as the formula $?p \to ?q$, we would want to translate (1-c) as the disjunction $(?p \to ?q) \vee (?r \to ?q)$. But again, this does not work well: while (1-c) may be true or false, the formula $(?p \to ?q) \vee (?r \to ?q)$ is a tautology.

Once more, one might want to blame this problem on the treatment of disjunction in InqB. Translating (1-c) by means of *inquisitive* disjunction, as $(?p \to ?q) \mathbin{\vee\!\!\!\vee} (?r \to ?q)$ fixes the problem, ensuring that (1-c) holds at a state only if one of its disjuncts does. But again, this would not work as a general account of disjunction: to establish that *Alice is at home or at school*, it is not necessary to establish in which of the two places she is, as would be required by treating disjunction as $\mathbin{\vee\!\!\!\vee}$. Thus, again, this strategy has the undesirable consequence that disjunctions in dependence statements need to be analyzed differently from disjunctions in other statements. By contrast, the proposal developed below shows that no departure from $\vee$ is needed to analyze (1-c). [2]

Next, consider the case of knowledge attributions, exemplified by (1-d). The epistemic extension of InqB, called *inquisitive epistemic logic* [8,1,3], comes with a simple analysis of the knowledge modality K_a. If $S_a \subseteq W \times W$ is a relation of epistemic accessibility for agent a, and $S_a[w] := \{v \in W \mid wS_av\}$, the truth-conditions for $K_a\varphi$ are as follows: [3]

$$M, w \models K_a\varphi \text{ iff } M, S_a[w] \models \varphi$$

This clause boils down to the standard epistemic logic clause when φ is truth-conditional, but it also allows us to deal with the case in which φ is a question.

---

[2] In dependence logic, disjunction is not defined via negation and conjunction, but taken as a primitive operator with the following support clause: $M, s \models \varphi \vee \psi$ iff $s = t_1 \cup t_2$ for some t_1, t_2 such that $M, t_1 \models \varphi$ and $M, t_2 \models \psi$. But this clause also renders $(?p \to ?q) \vee (?r \to ?q)$ a tautology. To see this, notice that any state s can be split into the states $s_q := s \cap |q|_M$ and $s_{\neg q} := s \cap |\neg q|_M$, and we have $M, s_q \models ?p \to ?q$ and $M, s_{\neg q} \models ?r \to ?q$. Thus, the problem with analyzing disjunctive dependence statements would not disappear if we treated disjunction by means of this clause. Thanks to an anonymous reviewer for pointing this out.

[3] In inquisitive epistemic logic, formulae $K_a\varphi$ are truth-conditional: the corresponding support clause just makes $K_a\varphi$ supported at s in case it is true at every world in s.

We can, e.g., analyze the statement *Alice knows whether* p directly as $K_a?p$, rather than having to paraphrase that statement as $K_a p \vee K_a \neg p$.

If we analyze (1-a) as $?p \to ?q$, we'd want to analyze (1-d) as $K_a(?p \to ?q)$. Once again, the result is not quite right. In inquisitive epistemic logic, this formula is equivalent to the following disjunction:

$$\bigvee_{\alpha \in S} K_a \alpha \qquad \text{where } S = \left\{ \begin{array}{ll} (p \to q) \wedge (\neg p \to q), & (p \to q) \wedge (\neg p \to \neg q) \\ (p \to \neg q) \wedge (\neg p \to q), & (p \to \neg q) \wedge (\neg p \to \neg q) \end{array} \right\}$$

This means that $K(?p \to ?q)$ is true only if Alice knows *how* the truth-value of q is determined by the one of p. But this is not required for the truth of (1-d): Alice may know that $?q$ is determined by $?p$ without knowing exactly how this dependency is realized; she might, e.g., be uncertain between two possibilities: (i) q is true if and only if p is; and (ii) q is true if and only if p is false.

Finally, consider the polar question (1-e) asking about the truth of (1-a). In inquisitive logic, if a statement is formalized by α, then the corresponding polar question is formalized by $?\alpha := \alpha \mathbin{\vee\!\!\!\vee} \neg\alpha$. Thus, if we formalize (1-a) as $?p \to ?q$ we would want to formalize (1-e) as $?(?p \to ?q)$. However, since $\neg(?p \to ?q) \equiv \bot$, we have $?(?p \to ?q) = (?p \to ?q) \mathbin{\vee\!\!\!\vee} \neg(?p \to ?q) \equiv ?p \to ?q$. Thus, the question in (1-e) would come out equivalent to the statement in (1-a)—clearly not the right result.

Summing up, then, in combination with the analysis of other logical items (negation, disjunction, knowledge, and the polar question operator) the assumption that (1-a) is translated as $?p \to ?q$ leads to blatantly wrong results.

4 Dependence statements are strict conditionals

All of the problems described in the previous section disappear if we formalize dependence statements not by means of the InqB-conditional $\to$, but rather by means of a generalized version of a strict conditional operator. To see how this solution works, the first step is to introduce such an operator into our logic. Thus, in this section we will work with the following language, denoted $\mathcal{L}^{\Rightarrow}$:[4]

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi \mid \varphi \to \varphi \mid \varphi \Rightarrow \varphi$$

The operators $\neg, \vee$, and $?$ are defined as in InqB; as before, we refer to $\mathbin{\vee\!\!\!\vee}$-free formulas as *classical formulas.* With respect to standard logics equipped with a strict conditional operator, the novelty lies in the fact that this operator can be applied to questions. Thus, our language contains, e.g., formulas like $?p \Rightarrow ?q$. These are precisely the formulas that I propose to regard as formal counterparts of dependence statements.

[4] For simplicity, here I consider a single strict conditional operator $\Rightarrow$. One could implement the same ideas in a multi-modal language, equipped with a family $\{\Rightarrow_a \mid a \in \mathcal{A}\}$ of strict conditional operators, associated with corresponding accessibility relations $\{R_a \mid a \in \mathcal{A}\}$. The indices $a \in \mathcal{A}$ may be viewed as different agents endowed with different information, or as different processes associated with different sets of possible outcomes. In this setting, different dependencies will hold at a world w from the perspective of different $a \in \mathcal{A}$. The completeness results established below all extend straightforwardly to this case.

A strict implication is a modal operator. To interpret it, we need to equip our set of worlds with a binary relation R which captures relative possibility. This leads us naturally to work with the standard structure of a *Kripke model*, i.e., a triple $M = \langle W, R, V\rangle$ where W and V are as above, and $R \subseteq W \times W$.

Intuitively, wRv means that v is *possible* at w in the relevant sense. E.g., in our initial examples we were thinking of a process that may yield different outcomes. In this context, wRv will mean that v is a possible outcome of the process as it takes place at w. But this is just one of many possible interpretations. One could also, e.g., take wRv to mean that v is compatible with the information available to an agent at w; in this case, the relevant dependencies will be epistemic in nature. As far as I can see, there are just as many different flavors of dependency as there are flavors of modality.

The semantics for the system $\mathsf{Inq}^{\Rightarrow}$ based on the language $\mathcal{L}^{\Rightarrow}$ is given in terms of support clauses relative to an information state $s \subseteq W$ in a Kripke model. The clauses for atoms and connectives are the same as in Definition 2.1. The support clause for $\Rightarrow$ is as follows.

Definition 4.1 [Support for $\Rightarrow$] $M, s \models \varphi \Rightarrow \psi$ iff $\forall w \in s : \varphi \models_{R[w]} \psi$

This clause makes the formula $\varphi \Rightarrow \psi$ truth-conditional, with the following truth-conditions.

Definition 4.2 [Truth-conditions for $\Rightarrow$] $M, w \models \varphi \Rightarrow \psi$ iff $\varphi \models_{R[w]} \psi$

That is, $\varphi \Rightarrow \psi$ is true at w in case φ entails ψ relative to the information state $R[w]$. Making the contextual entailment relation explicit, this becomes:

- $M, w \models \varphi \Rightarrow \psi$ iff $\forall t \subseteq R[w] : M, t \models \varphi$ implies $M, t \models \psi$

To understand what this clause delivers, let us first look at the special case in which $\Rightarrow$ applies to two truth-conditional formulas. In this case, it is not hard to see that we recover the standard clause for the strict conditional operator.

Proposition 4.3 (Strict conditional recovered)
If α and β are truth-conditional, then $M, w \models \alpha \Rightarrow \beta$ iff $R[w] \cap |\alpha|_M \subseteq |\beta|_M$

It is also easy to check that all classical formulas are truth-conditional. This means that, when we restrict ourselves to the $\vee\!\!\!\vee$-free fragment of $\mathsf{Inq}^{\Rightarrow}$, what we have is just (a support-based implementation of) classical propositional logic augmented with a standard strict conditional operator.

The novel feature of $\mathsf{Inq}^{\Rightarrow}$ lies in the fact that our language contains not only statements, but also questions, and $\Rightarrow$ can be applied meaningfully to questions μ and ν to yield a modal statement $\mu \Rightarrow \nu$. What does this statement express? By Definition 4.2, $\mu \Rightarrow \nu$ is true at w just in case $\mu \models_{R[w]} \nu$. As we discussed in Section 2, the relation $\mu \models_{R[w]} \nu$ captures the fact that question μ determines question ν relative to the state $R[w]$. Thus, $\mu \Rightarrow \nu$ is a truth-conditional formula which is true or false at a world w according to whether μ determines ν in the associated set of worlds $R[w]$. This, I propose, is the right way to understand the semantics of dependence statements: a statement of the form *question μ determines question ν* should be rendered in our formal language not as the InqB-conditional $\mu \to \nu$, but as the strict conditional $\mu \Rightarrow \nu$. For instance, our dependence statement in (1-a) should be formalized as $?p \Rightarrow ?q$. This formula is true or false at w depending on whether the truth-value of q is functionally determined by the truth-value of p throughout the set $R[w]$.

$$M, w \models ?p \Rightarrow ?q \text{ iff } \exists f : \{0,1\} \to \{0,1\} \text{ s.t. } \forall v \in R[w] : V(v,q) = f(V(v,p))$$

Notice that $?p \Rightarrow ?q$ may well be false at a world w; so, the first problem we discussed in the previous section does not arise. Also, $?p \Rightarrow ?q$ is truth-conditional. Thus, dependence statements are translated by formulae that have the fundamental semantic property that we associate with statements, namely, truth-conditionality. This also ensures that embeddings under other operators work just as well for dependence statements as for ordinary statements.

Consider negation: unlike $\neg(?p \to ?q)$, the formula $\neg(?p \Rightarrow ?q)$ is consistent, and it is true at w just in case $?p \Rightarrow ?q$ is false, i.e., just in case $?p \not\models_{R[w]} ?q$. Thus, we get a good analysis of (1-b) as a negation of a dependence statement.

Consider disjunction: unlike $(?p \to ?q) \vee (?r \to ?q)$, the formula $(?p \Rightarrow ?q) \vee (?r \Rightarrow ?q)$ is not a tautology: it is a truth-conditional formula which is true at w just in case one of its disjuncts is true, i.e., in case $?p \models_{R[w]} ?q$ or $?r \models_{R[w]} ?q$. Thus we get a good analysis for (1-c) as a disjunction of dependence statements.

Next consider the embedding of a dependence statement $?p \Rightarrow ?q$ in the scope of a knowledge modality K_a. We get:

$$\begin{aligned} M, w \models K_a(?p \Rightarrow ?q) \quad &\text{iff} \quad M, S[w] \models ?p \Rightarrow ?q \\ &\text{iff} \quad \forall v \in S[w] : M, v \models ?p \Rightarrow ?q \\ &\text{iff} \quad \forall v \in S[w] : \exists f_v : \{0,1\} \to \{0,1\} \text{ such that} \\ &\qquad\qquad \forall u \in R[v] : V(u,q) = f_v(V(u,p)) \end{aligned}$$

Thus, $K_a(?p \Rightarrow ?q)$ is true at a world w in case $?p \Rightarrow ?q$ is true in all worlds compatible with the agent's knowledge at w. This means that for all worlds v compatible with the agent's knowledge we have a corresponding functional dependency that holds across $R[v]$. But different worlds $v, v' \in S[w]$ might be associated with different sets $R[v], R[v']$ of possibilities, in which the dependency of $?q$ on $?p$ might be realized by different functions $f_v, f_{v'}$. Although there might be a single dependence function f which works on $R[v]$ for all worlds $v \in S[w]$, there need not be. Thus, as we expect, the formula $K_a(?p \Rightarrow ?q)$ does not entail the following disjunction (although it is entailed by it):

$$\bigvee_{\alpha \in S} K_a \alpha \qquad \text{where } S = \left\{ \begin{array}{ll} (p \Rightarrow q) \wedge (\neg p \Rightarrow q), & (p \Rightarrow q) \wedge (\neg p \Rightarrow \neg q) \\ (p \Rightarrow \neg q) \wedge (\neg p \Rightarrow q), & (p \Rightarrow \neg q) \wedge (\neg p \Rightarrow \neg q) \end{array} \right\}$$

Thus, by combining the proposed analysis of dependence statements with the inquisitive analysis of the knowledge modality we obtain a good analysis of the knowledge attribution in (1-d)—one that predicts that knowing that a dependency holds does not require knowing how it is realized.

Finally, take the question (1-e): while $?(?p \to ?q)$ is equivalent to $?p \to ?q$, the formula $?(?p \Rightarrow ?q)$ is not equivalent to $?p \Rightarrow ?q$. The latter is a statement, while $?(?p \Rightarrow ?q)$ is a polar question that can be resolved in two different ways, namely, by establishing that $?p \Rightarrow ?q$ is true, or by establishing that it is false:

$$M, s \models ?(?p \Rightarrow ?q) \text{ iff } s \subseteq |?p \Rightarrow ?q|_M \text{ or } s \subseteq |\neg(?p \Rightarrow ?q)|_M$$

So, we get a good analysis of (1-e) as a polar question about the statement (1-a).

In conclusion, the proposed analysis, in addition to having a *prima facie* intuitive appeal, interacts well with the rest of the logical repertoire. When we analyze dependence statements as strict conditionals involving questions, the problems discussed in the previous section disappear. The fact that all these pieces fall naturally into place, I submit, provides a good case for the proposal, and motivates an investigation of the logic of strict conditionals in the inquisitive setting. To this I turn in the next section.

5 The inquisitive logic of strict conditionals

5.1 Declaratives and normal form

In inquisitive logic, entailment is defined naturally as preservation of support:

- $\Phi \models \psi$ iff $\forall M, s$: if $M, s \models \varphi$ for all $\varphi \in \Phi$, then $M, s \models \psi$

Logical equivalence, denoted $\equiv$, is defined as support at the same states in all models, and coincides with mutual entailment.

Following a standard strategy in the study of inquisitive logics [1,2,3,4], we first isolate a fragment of $\mathcal{L}^{\Rightarrow}$ consisting only of truth-conditional formulas. The set of *declaratives*, $\mathcal{L}_!$, is given by the following BNF, where $\varphi \in \mathcal{L}^{\Rightarrow}$:

$$\alpha ::= p \mid \bot \mid \alpha \wedge \alpha \mid \varphi \to \alpha \mid \varphi \Rightarrow \varphi$$

In words, a formula α is a declarative iff the only occurrences of $\vee\!\!\!\vee$ in α, if any, are within the scope of a strict conditional or in the antecedent of $\to$. It is then straightforward to show the following.

Proposition 5.1 *All $\alpha \in \mathcal{L}_!$ are truth-conditional.*

As in InqB (see [3,4]), so also in $\mathsf{Inq}^{\Rightarrow}$ we can associate to each formula φ a set $\mathcal{R}(\varphi)$ of declaratives, called the resolutions of φ, which jointly characterize φ.

Definition 5.2 [Resolutions]

- $\mathcal{R}(\varphi) = \{\varphi\}$ if φ is an atom, $\bot$, or a strict conditional;
- $\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\varphi) \text{ and } \beta \in \mathcal{R}(\psi)\}$
- $\mathcal{R}(\varphi \vee\!\!\!\vee \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$
- $\mathcal{R}(\varphi \to \psi) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)}(\alpha \to f(\alpha)) \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$

It is easy to see by induction that a declarative is the only resolution of itself.

Remark 5.3 For all $\alpha \in \mathcal{L}_!$, $\mathcal{R}(\alpha) = \{\alpha\}$.

On the other hand, if φ is a question then the elements of $\mathcal{R}(\varphi)$ can be thought of as syntactically generated answers to φ. For instance, $\mathcal{R}(?p) = \{p, \neg p\}$. An easy induction suffices to establish the following normal form result.

Definition 5.4 [Inquisitive normal form] For all $\varphi \in \mathcal{L}^{\Rightarrow}$, $\varphi \equiv \bigvee\!\!\!\bigvee \mathcal{R}(\varphi)$.

5.2 Proof system

In this section we describe a Hilbert-style proof-system for the logic $\mathsf{Inq}^{\Rightarrow}$. For convenience, we divide the axioms into a set of propositional axioms and a set of axioms which involve specifically the strict conditional operator.

The propositional axioms, inherited from the propositional logic InqB, are all instances of the following schemata, where $\varphi, \psi, \chi \in \mathcal{L}^{\Rightarrow}$ and $\alpha \in \mathcal{L}_!$:

(i) $\varphi \to (\psi \to \varphi)$

(ii) $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

(iii) $\varphi \to (\psi \to \varphi \wedge \psi)$

(iv) $\varphi \wedge \psi \to \varphi, \quad \varphi \wedge \psi \to \psi$

(v) $\varphi \to \varphi \vee\!\!\!\vee \psi, \quad \psi \to \varphi \vee\!\!\!\vee \psi$

(vi) $(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee\!\!\!\vee \psi \to \chi))$

(vii) $\bot \to \varphi$

(viii) $\neg\neg\alpha \to \alpha$

(ix) $(\alpha \to \varphi \vee\!\!\!\vee \psi) \to (\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi)$

Axioms (i)–(vii) are essentially the axioms of intuitionistic propositional logic with $\vee\!\!\!\vee$ in the role of intuitionistic disjunction (see [7] for discussion). Axiom (viii) captures the fact that declaratives are truth-conditional and, as a consequence, obey classical logic. Axiom (ix) captures the fact that, again due to truth-conditionality, declaratives distribute over inquisitive disjunctions. [5]

The axioms for the strict implication modality are all instances of the following schemata, where φ, ψ, χ stand for arbitrary formulas and α for a declarative:

(x) transitivity: $(\varphi \Rightarrow \psi) \to ((\psi \Rightarrow \chi) \to (\varphi \Rightarrow \chi))$

(xi) import-export: $(\varphi \wedge \psi \Rightarrow \chi) \leftrightarrow (\varphi \Rightarrow (\psi \to \chi))$

(xii) $\Rightarrow$-split: $(\alpha \Rightarrow \varphi \vee\!\!\!\vee \psi) \to (\alpha \Rightarrow \varphi) \vee (\alpha \Rightarrow \psi)$

The inference rules are modus ponens and conditional necessitation:

$$\frac{\varphi \to \psi \quad \varphi}{\psi}\ (\mathsf{MP}) \qquad\qquad \frac{\varphi \to \psi}{\varphi \Rightarrow \psi}\ (\mathsf{CN})$$

We write $\vdash \psi$ if the formula ψ is derivable in this system, $\varphi \dashv\vdash \psi$ if $\varphi \leftrightarrow \psi$ is derivable, and $\Phi \vdash \psi$ if $\varphi_1 \wedge \cdots \wedge \varphi_n \to \psi$ is derivable for some $\varphi_1, \ldots, \varphi_n \in \Phi$.

It is easy to check directly that all instances of the axioms are valid, and that the inference rules preserves validity, which implies that the proof system is sound. In the remainder of this section we show that it is also complete.

---

[5] The first axiomatization given for InqB [7] used slightly different axioms: double negation elimination was restricted to atoms, and instead of the $\vee\!\!\!\vee$-split axiom it included the Kreisel-Putnam axiom $(\neg\varphi \to \psi \vee\!\!\!\vee \chi) \to (\neg\varphi \to \psi) \vee\!\!\!\vee (\neg\varphi \to \chi)$ for arbitrary φ, ψ, χ. In the setting of $\mathsf{Inq}^{\Rightarrow}$, having double negation elimination for atoms only is not enough: we must minimally have it for strict conditionals as well. On the other hand, the choice between $\vee\!\!\!\vee$-split and the KP axiom is a matter of convenience. Using $\vee\!\!\!\vee$-split has an advantage in terms of generality, since this axiom also plays a role in non-classical versions of inquisitive logic [16,17,6].

As usual, completeness is established by constructing a canonical model. To show that the construction works, we first need a few results about our proof system. One thing that will play a crucial role is that the system has enough resources to justify the normal form result given by Proposition 5.4.

Lemma 5.5 $\varphi \dashv\vdash \bigvee\!\!\!\!\bigvee \mathcal{R}(\varphi)$

Proof. By induction on the structure of φ. The only step that requires some work is the one for $\varphi = (\psi \to \chi)$. But the proof has nothing to do with strict implication: it is essentially the same as for InqB (see Lemma 3.3.4 in [3]). □

The following two lemmata connecting provability of/from a formula to provability of/from its resolutions will play a crucial role in the completeness proof.

Lemma 5.6 *If* $\Gamma \subseteq \mathcal{L}_!$ *and* $\Gamma \vdash \varphi$*, then* $\Gamma \vdash \alpha$ *for some* $\alpha \in \mathcal{R}(\varphi)$

Proof. First we show the following claim:

$$\text{If } \vdash \varphi, \text{ then } \vdash \alpha \text{ for some } \alpha \in \mathcal{R}(\varphi) \tag{1}$$

We show (1) by induction on the length of the shortest proof of φ. If φ is provable with a proof of length 1, then φ is an axiom. Suppose φ is a propositional axiom: then it is straightforward to check, considering each case in turn, that a resolution of φ is a classical tautology. By way of example, suppose φ is an instance $\psi \to (\chi \to \psi)$ of axiom (i); the following tautology is a resolution of φ:

$$\bigwedge_{\alpha \in \mathcal{R}(\psi)} (\alpha \to \bigwedge_{\beta \in \mathcal{R}(\chi)} (\beta \to \alpha))$$

But in restriction to declaratives, our system contains a complete set of axioms for classical propositional logic, and so it proves all classical tautologies.

Next, suppose φ is one of the axioms for the strict conditional operator. All instances of such axioms are declaratives. Thus, by Remark 5.3, we have $\mathcal{R}(\varphi) = \{\varphi\}$; since φ is provable, obviously a resolution of φ is provable.

Now consider the inductive case. We have only two cases to consider:

(i) φ is obtained by modus ponens from formulas $\psi \to \varphi$ and ψ which are provable with shorter proofs. By induction hypothesis, the system proves a resolution γ of $\psi \to \varphi$ and a resolution β_0 of ψ. By definition of resolutions of an implication, γ has the form $\bigwedge_{\beta \in \mathcal{R}(\psi)} (\beta \to f(\beta))$ for some $f : \mathcal{R}(\psi) \to \mathcal{R}(\varphi)$. But then, clearly, $f(\beta_0)$ is a provable resolution of φ.

(ii) $\varphi = (\psi \Rightarrow \chi)$ is obtained by modal necessitation from a formula $\psi \to \chi$ that has a shorter proof. In this case, φ is a declarative; so, by Remark 5.3, $\mathcal{R}(\varphi) = \{\varphi\}$. Thus, our proof of φ is also a proof of a resolution of φ.

This completes the inductive proof of (1). To prove Lemma 5.6, suppose $\Gamma \subseteq \mathcal{L}_!$ and $\Gamma \vdash \varphi$. This means that $\vdash \gamma_1 \wedge \cdots \wedge \gamma_n \to \varphi$ for some $\gamma_1, \ldots, \gamma_n \in \Gamma$. Let $\gamma := \gamma_1 \wedge \cdots \wedge \gamma_n$: since γ is a declarative, by Remark 5.3 we have $\mathcal{R}(\gamma) = \{\gamma\}$. Thus, by definition of resolutions of an implication, we have:

$$\mathcal{R}(\gamma \to \varphi) = \{\gamma \to f(\gamma) \mid f : \{\gamma\} \to \mathcal{R}(\varphi)\} = \{\gamma \to \alpha \mid \alpha \in \mathcal{R}(\varphi)\}$$

Since $\vdash \gamma \to \varphi$, by (1) it follows that $\vdash \gamma \to \alpha$ for some resolution $\alpha \in \mathcal{R}(\varphi)$. Since obviously $\Gamma \vdash \gamma$, we can conclude that $\Gamma \vdash \alpha$. $\Box$

Lemma 5.7 *If $\Phi, \chi \nvdash \psi$, then $\Phi, \alpha \nvdash \psi$ for some $\alpha \in \mathcal{R}(\chi)$*

Proof. By contraposition, suppose $\Phi, \alpha \vdash \psi$ for all $\alpha \in \mathcal{R}(\chi)$. Using the axioms for $\mathbin{\vee\!\!\!\vee}$ we infer that $\Phi, \mathbin{\vee\!\!\!\vee}\mathcal{R}(\chi) \vdash \psi$. By Lemma 5.5 we have $\Phi, \chi \vdash \psi$.$\Box$

Using this lemma inductively, we obtain the following corollary (for the details, see the analogous result in the propositional setting, Lemma 3.3.7 in [3]).

Corollary 5.8 *If $\Phi \nvdash \psi$, then there is a set of declaratives $\Gamma \subseteq \mathcal{L}_!$ which contains a resolution of each $\varphi \in \Phi$ and such that $\Gamma \nvdash \psi$.*

In addition to these results, in the completeness proof we will use some provable features of strict conditionals, spelled out in the following lemmata.

Lemma 5.9 *If $\varphi' \vdash \varphi$ and $\psi \vdash \psi'$ then $\varphi \Rightarrow \psi \vdash \varphi' \Rightarrow \psi'$.*

Proof. From $\varphi' \vdash \varphi$ and $\psi \vdash \psi'$ we have $\vdash \varphi' \Rightarrow \varphi$ and $\vdash \psi \Rightarrow \psi'$ by conditional necessitation. By transitivity we get $\vdash (\varphi \Rightarrow \psi) \to (\varphi' \Rightarrow \psi')$. $\Box$

As an immediate corollary we get replacement of provably equivalent formulas.

Corollary 5.10 *If $\varphi \dashv\vdash \varphi'$ and $\psi \dashv\vdash \psi'$ then $\varphi \Rightarrow \psi \dashv\vdash \varphi' \Rightarrow \psi'$*

Another provable feature of $\Rightarrow$ concerns the behavior of $\mathbin{\vee\!\!\!\vee}$ in antecedents.

Lemma 5.11 *For any $\varphi, \psi, \chi \in \mathcal{L}^{\Rightarrow}$, $(\varphi \mathbin{\vee\!\!\!\vee} \psi \Rightarrow \chi) \dashv\vdash (\varphi \Rightarrow \chi) \wedge (\psi \Rightarrow \chi)$.*

Proof. By Corollary 5.10 and the import-export axiom we have that $\varphi \Rightarrow \chi \dashv\vdash (\top \wedge \varphi) \Rightarrow \chi \dashv\vdash \top \Rightarrow (\varphi \to \chi)$. Analogously we have $\psi \Rightarrow \chi \dashv\vdash \top \Rightarrow (\psi \to \chi)$. Thus, $(\varphi \Rightarrow \chi) \wedge (\psi \Rightarrow \psi) \dashv\vdash (\top \Rightarrow (\varphi \to \chi)) \wedge (\top \Rightarrow (\psi \to \chi))$.

We have $(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \mathbin{\vee\!\!\!\vee} \psi \to \chi))$ as an axiom. Using this, by conditional necessitation, transitivity, and import-export, we can show that $(\top \Rightarrow (\varphi \to \chi)) \wedge (\top \Rightarrow (\psi \to \chi)) \vdash \top \Rightarrow (\varphi \mathbin{\vee\!\!\!\vee} \psi \to \chi)$. By import-export and Cor. 5.10, $\top \Rightarrow (\varphi \mathbin{\vee\!\!\!\vee} \psi \to \chi) \dashv\vdash (\top \wedge (\varphi \mathbin{\vee\!\!\!\vee} \psi)) \Rightarrow \chi \dashv\vdash \varphi \mathbin{\vee\!\!\!\vee} \psi \Rightarrow \chi$. This shows the right-to-left direction. The converse direction follows from the provable monotonicity of $\Rightarrow$ on the left (Lemma 5.9) since $\varphi \vdash \varphi \mathbin{\vee\!\!\!\vee} \psi$ and $\psi \vdash \varphi \mathbin{\vee\!\!\!\vee} \psi$. $\Box$

Finally, the next lemma connects strict conditionals applying to arbitrary formulas to Boolean combinations of strict conditionals applying to declaratives.

Lemma 5.12 $\varphi \Rightarrow \psi \dashv\vdash \bigvee_{f:\mathcal{R}(\varphi)\to\mathcal{R}(\psi)} \bigwedge_{\alpha\in\mathcal{R}(\varphi)}(\alpha \Rightarrow f(\alpha))$

The proof of this lemma uses the following auxiliary result.

Lemma 5.13 *Let $\alpha, \beta, \gamma \in \mathcal{L}_!$. If $\alpha \vdash \gamma$ and $\beta \vdash \gamma$ then $\alpha \vee \beta \vdash \gamma$.*

Proof. This follows from the fact that, restricted to declaratives, our system includes a complete set of axioms for classical propositional logic. $\Box$

Proof of Lemma 5.12. Let $\alpha \in \mathcal{R}(\varphi)$. We have $\alpha \Rightarrow \psi \dashv\vdash \alpha \Rightarrow \mathbin{\vee\!\!\!\vee}\mathcal{R}(\psi)$ by Lemma 5.5 and Cor. 5.10. We also have $\alpha \Rightarrow \mathbin{\vee\!\!\!\vee}\mathcal{R}(\psi) \dashv\vdash \bigvee_{\beta\in\mathcal{R}(\psi)}(\alpha \Rightarrow \beta)$: the left-to-right direction uses the $\Rightarrow$-split axiom; the converse uses Lemma

5.13 once we notice that for each $\beta \in \mathcal{R}(\psi)$, $\alpha \Rightarrow \beta \vdash \alpha \Rightarrow \bigvee\!\!\bigvee \mathcal{R}(\psi)$ by Lemma 5.9. As a consequence, we have:

$$\bigwedge_{\alpha \in \mathcal{R}(\varphi)} (\alpha \Rightarrow \psi) \dashv\vdash \bigwedge_{\alpha \in \mathcal{R}(\varphi)} \bigvee_{\beta \in \mathcal{R}(\psi)} (\alpha \Rightarrow \beta) \qquad (2)$$

By Lemma 5.11 we have $\bigwedge_{\alpha \in \mathcal{R}(\varphi)} (\alpha \Rightarrow \psi) \dashv\vdash \bigvee\!\!\bigvee \mathcal{R}(\varphi) \Rightarrow \psi$. By Lemma 5.5 and Corollary 5.10, the latter is provably equivalent to $\varphi \Rightarrow \psi$. Thus, the left-hand-side of (2) is provably equivalent to $\varphi \Rightarrow \psi$. As for the right hand side, recall that in restriction to declaratives, our system contains a complete set of axioms for classical propositional logic. In particular, we can distribute $\wedge$ over $\vee$. Using this we can show that the right-hand-side of (2) is provably equivalent to $\bigvee_{f:\mathcal{R}(\varphi) \to \mathcal{R}(\psi)} \bigwedge_{\alpha \in \mathcal{R}(\varphi)} (\alpha \Rightarrow f(\alpha))$. □

5.3 Completeness by canonical model

We call a set $\Gamma \subseteq \mathcal{L}_!$ a *declarative theory* if for all $\alpha \in \mathcal{L}_!$, $\Gamma \vdash \alpha$ only if $\alpha \in \Gamma$. We call Γ a *complete declarative theory* if it is a declarative theory and for every $\alpha \in \mathcal{L}_!$ it contains exactly one of α and $\neg\alpha$ (which implies consistency).

Definition 5.14 [Canonical model]
The canonical model for $\mathsf{Inq}^{\Rightarrow}$ is the model $M^c = \langle W^c, R^c, V^c \rangle$ where:

- W^c is the set of all complete declarative theories;
- $\Gamma R^c \Gamma'$ iff $\forall \alpha, \beta \in \mathcal{L}_! : (\alpha \Rightarrow \beta) \in \Gamma$ and $\alpha \in \Gamma'$ implies $\beta \in \Gamma'$
- $V^c(\Gamma, p) = 1$ iff $p \in \Gamma$

The following lemmata about complete declarative theories are mostly familiar from classical propositional and modal logic; I omit the straightforward proofs.

Lemma 5.15 *If $\Gamma \in W^c$ and $\alpha \vee \beta \in \Gamma$, then $\alpha \in \Gamma$ or $\beta \in \Gamma$.*

Lemma 5.16 *If $S \subseteq W^c$, then $\bigcap S$ is a declarative theory.*

Lemma 5.17 *If $\Delta \subseteq \mathcal{L}_!$ is consistent then $\Delta \subseteq \Gamma$ for some $\Gamma \in W^c$.*

The next two lemmata establish important connections between provability and the structure of the canonical model.

Lemma 5.18 *Let $S \subseteq W^c$. If $\bigcap S \nvdash \varphi \to \psi$, then for some subset $T \subseteq S$ we have $\bigcap T \vdash \varphi$ and $\bigcap T \nvdash \psi$.*

Proof. If $\bigcap S \nvdash \varphi \to \psi$, then $\bigcap S, \varphi \nvdash \psi$. By Lemma 5.7 there is some $\alpha \in \mathcal{R}(\varphi)$ such that $\bigcap S, \alpha \nvdash \psi$. Let $T := \{\Gamma \in S \mid \alpha \in \Gamma\}$. Since $\alpha \in \bigcap T$, by Lemma 5.5 we have $\bigcap T \vdash \varphi$. It remains to be seen that $\bigcap T \nvdash \psi$.

Towards a contradiction, suppose $\bigcap T \vdash \psi$. By Lemma 5.6 this implies $\bigcap T \vdash \beta$ for some $\beta \in \mathcal{R}(\psi)$. By Lemma 5.16 we have $\beta \in \bigcap T$. Thus, for all $\Gamma \in T$ we have $\beta \in \Gamma$, and so also $\alpha \to \beta \in \Gamma$. But now consider any $\Gamma \in S - T$. Since $\alpha \notin \Gamma$, by completeness we have $\neg\alpha \in \Gamma$ and so also $\alpha \to \beta \in \Gamma$. We have thus reached the conclusion that $\alpha \to \beta$ belongs to all $\Gamma \in S$, no matter whether $\Gamma \in T$ or $\Gamma \in S - T$. Thus, $\alpha \to \beta \in \bigcap S$. But this implies that $\bigcap S, \alpha \vdash \beta$, which by Lemma 5.5 gives $\bigcap S, \alpha \vdash \psi$, contrary to assumption. □

Lemma 5.19 *Let $\Gamma \in W^c$. If $(\varphi \Rightarrow \psi) \notin \Gamma$ then $\bigcap R^c[\Gamma] \nvdash \varphi \to \psi$.*

Proof. Suppose $(\varphi \Rightarrow \psi) \notin \Gamma$. By Equation (2) above, this means that Γ does not contain $\bigwedge_{\alpha \in \mathcal{R}(\varphi)} \bigvee_{\beta \in \mathcal{R}(\psi)} (\alpha \Rightarrow \beta)$. This means that there is some $\alpha_j \in \mathcal{R}(\varphi)$ such that for all $\beta \in \mathcal{R}(\psi)$, $\alpha_j \Rightarrow \beta \notin \Gamma$. Now let $\mathcal{R}(\psi) = \{\beta_1, \dots, \beta_n\}$. For $i \leq n$ set $\Delta_i := \{\gamma \in \mathcal{L}_! \mid (\alpha_j \Rightarrow \gamma) \in \Gamma\} \cup \{\neg\beta_i\}$. We claim that Δ_i is consistent. Towards a contradiction, suppose not. That means that there are $\gamma_1, \dots, \gamma_n$ such that $\alpha \Rightarrow \gamma_k \in \Gamma$ for each $k \leq n$ and $\gamma_1, \dots, \gamma_n, \neg\beta_i \vdash \bot$. Since $\neg\beta_i$ is a declarative and for declaratives we have all the axioms of classical propositional logic, we have $\gamma_1, \dots, \gamma_n \vdash \beta_i$. By conditional necessitation and transitivity we easily obtain $\alpha_j \Rightarrow \gamma_1, \dots, \alpha_j \Rightarrow \gamma_n \vdash \alpha_j \Rightarrow \beta_i$. Since all the formulas $\alpha_j \Rightarrow \gamma_i$ are in Γ, it follows that $\alpha_j \Rightarrow \beta_i \in \Gamma$, contrary to assumption.

Thus, Δ_i is indeed consistent. This means, by Lemma 5.17, that it can be extended to a $\Gamma_i \in W^c$. We claim that $\Gamma R^c \Gamma_i$. To see this, let $\gamma, \delta \in \mathcal{L}_!$ and suppose that $\gamma \Rightarrow \delta \in \Gamma$ and $\gamma \in \Gamma_i$. We need to show that $\delta \in \Gamma_i$.

By Lemma 5.9, $\gamma \Rightarrow \delta \vdash \alpha_j \wedge \gamma \Rightarrow \delta$. By the import-export axiom we have $\alpha_j \wedge \gamma \Rightarrow \delta \dashv\vdash \alpha_j \Rightarrow (\gamma \to \delta)$. Since $\gamma \Rightarrow \delta \in \Gamma$, also $\alpha_j \Rightarrow (\gamma \to \delta) \in \Gamma$, and so by definition of Δ_i we have $\gamma \to \delta \in \Delta_i \subseteq \Gamma_i$. Since Γ_i also contains γ, it follows that Γ_i contains δ, as we needed to show. Thus $\Gamma R^c \Gamma_i$.

Towards a contradiction, suppose that $\bigcap R^c[\Gamma] \vdash \varphi \to \psi$. By Lemma 5.6 and Lemma 5.15, for some $f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)$ we have $\bigcap R^c[\Gamma] \vdash \bigwedge_{\alpha \in \mathcal{R}(\varphi)} (\alpha \Rightarrow f(\alpha))$. In particular, $\bigcap R^c[\Gamma] \vdash \alpha_j \Rightarrow f(\alpha_j)$. Suppose $f(\alpha_j) = \beta_i$, so that $\bigcap R^c[\Gamma] \vdash \alpha_j \to \beta_i$. By Lemma 5.16 we obtain $(\alpha_j \to \beta_i) \in \bigcap R^c[\Gamma]$. Since $\Gamma_i \in R^c[\Gamma]$ we have $(\alpha_j \to \beta_i) \in \Gamma_i$. Since $\alpha_j \Rightarrow \alpha_j$ is provable, we have $(\alpha_j \Rightarrow \alpha_j) \in \Gamma$, and so $\alpha_j \in \Gamma_i$ by construction of Γ_i. Since Γ_i contains both $\alpha_j \to \beta_i$ and α_j, it follows that $\beta_i \in \Gamma_i$. But this is impossible, since by construction $\neg\beta_i \in \Gamma_i$ and Γ_i is consistent. Thus, $\bigcap R^c[\Gamma] \nvdash \varphi \to \psi$. □

With the help of these lemmata, the bridge between provability and semantics in the canonical model can be built. Usually, semantics is based on truth-conditions, and so the relevant bridge takes the form of a *truth lemma*, equating truth at a world with provability from it. In our setting, the fundamental semantic notion is not truth at a world, but support at an information state. Accordingly, the relevant bridge takes the form of a *support lemma*, which equates support at a state S in M^c with provability from the intersection of the theories in S (if $S = \emptyset$ we let $\bigcap S := \mathcal{L}_!$).

Lemma 5.20 (Support lemma)
For every $S \subseteq W^c$ and $\varphi \in \mathcal{L}^{\Rightarrow}$: $M^c, S \models \varphi$ iff $\bigcap S \vdash \varphi$.

Proof. By induction on the structure of φ. The only interesting cases are the inductive steps for the two implications $\to$ and $\Rightarrow$.

- $\varphi = (\psi \to \chi)$. Suppose $\bigcap S \vdash \psi \to \chi$. Take any $T \subseteq S$ with $M^c, T \models \psi$. By induction hypothesis we have $\bigcap T \vdash \psi$. Since $T \subseteq S$, $\bigcap T \supseteq \bigcap S$, so $\bigcap T \vdash \psi \to \chi$. By modus ponens it follows that $\bigcap T \vdash \chi$, which by induction hypothesis gives $M^c, T \models \chi$. This shows that $M^c, S \models \psi \to \chi$.

  For the converse direction, suppose $\bigcap S \nvdash \psi \to \chi$. By Lemma 5.18 there is a subset $T \subseteq S$ such that $\bigcap T \vdash \psi$ but $\bigcap T \nvdash \chi$. By induction hypothesis, this translates to $M^c, T \models \psi$ but $M^c, T \not\models \chi$, which implies $M^c, S \not\models \psi \to \chi$.

- $\varphi = (\psi \Rightarrow \chi)$. Suppose $\bigcap S \vdash \psi \Rightarrow \chi$. Since $\psi \Rightarrow \chi$ is a declarative, by Lemma 5.16 we have $(\psi \Rightarrow \chi) \in \bigcap S$. Now take any world $\Gamma \in S$ and take any $T \subseteq R[\Gamma]$ such that $M^c, T \models \psi$. We want to show that $M^c, T \models \chi$.

  Since $\Gamma \supseteq \bigcap S$, we have $\psi \Rightarrow \chi \in \Gamma$. Since Γ is a declarative theory, by Lemma 5.12, it must contain $\bigvee_{f:\mathcal{R}(\psi)\to\mathcal{R}(\chi)} \bigwedge_{\alpha\in\mathcal{R}(\psi)}(\alpha \Rightarrow f(\alpha))$. By Lemma 5.15 , Γ must then contain $\bigwedge_{\alpha\in\mathcal{R}(\psi)}(\alpha \Rightarrow f(\alpha))$ for some $f : \mathcal{R}(\psi) \to \mathcal{R}(\chi)$.

  Since $M^c, T \models \psi$, the induction hypothesis gives $\bigcap T \vdash \psi$. By Lemma 5.6 we have $\bigcap T \vdash \alpha_i$ for some $\alpha_i \in \mathcal{R}(\psi)$. By Lemma 5.16, this implies $\alpha_i \in \bigcap T$, and so $\alpha_i \in \Gamma'$ for all $\Gamma' \in T$. Now take any $\Gamma' \in T$: since $\Gamma R^c \Gamma'$, and since $\alpha_i \Rightarrow f(\alpha_i) \in \Gamma$ and $\alpha_i \in \Gamma'$, we must have $f(\alpha_i) \in \Gamma'$. This holds for all $\Gamma' \in T$, and so $\bigcap T \vdash f(\alpha_i)$. Since $f(\alpha_i) \in \mathcal{R}(\chi)$, it follows from Lemma 5.5 that $\bigcap T \vdash \chi$. By induction hypothesis, this gives $M^c, T \models \chi$.

  Summing up, we have shown that for all $\Gamma \in S$ and $T \subseteq R[\Gamma]$, $M^c, T \models \psi$ implies $M^c, T \models \chi$. This is just what is required to have $M^c, S \models \psi \Rightarrow \chi$.

  For the converse direction, suppose $\bigcap S \nvdash \psi \Rightarrow \chi$. Then $(\psi \Rightarrow \chi) \notin \Gamma$ for some $\Gamma \in S$. By Lemma 5.19 it follows that $\bigcap R^c[\Gamma] \nvdash \psi \to \chi$. Then by Lemma 5.18 there is some $T \subseteq R^c[\Gamma]$ with $\bigcap T \vdash \psi$ and $\bigcap T \nvdash \chi$. By the induction hypothesis, this translates to $M^c, T \models \varphi$ and $M^c, T \not\models \psi$. Thus, for some $\Gamma \in S$ and some $T \subseteq R^c[\Gamma]$ we have $M^c, \Gamma \models \psi$ and $M^c, \Gamma \not\models \chi$, which shows that $M^c, S \not\models \psi \Rightarrow \chi$. □

Finally, we can use the canonical model to prove our completeness result. [6]

Theorem 5.21 *For all* $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\Rightarrow}$, $\Phi \models \psi$ *iff* $\Phi \vdash \psi$.

Proof. Suppose $\Phi \nvdash \psi$. By Lemma 5.8, there is a set $\Delta \subseteq \mathcal{L}_!$ that contains a resolution of each $\varphi \in \Phi$ and such that $\Delta \nvdash \psi$. Let $S_\Delta := \{\Gamma \in W^c \mid \Delta \subseteq \Gamma\}$. We claim that (i) $M^c, S_\Delta \models \varphi$ for all $\varphi \in \Phi$, but (ii) $M^c, S_\Delta \not\models \psi$.

To show (i), take $\varphi \in \Phi$. For some $\alpha \in \mathcal{R}(\varphi)$ we have $\alpha \in \Delta \subseteq \bigcap S_\Delta$. The support lemma gives $M^c, S_\Delta \models \alpha$; by Proposition 5.4 we then get $M^c, S_\Delta \models \varphi$.

For (ii), suppose towards a contradiction that $M^c, S_\Delta \models \psi$. Proposition 5.4 implies that $M^c, S_\Delta \models \beta$ for some $\beta \in \mathcal{R}(\psi)$. By the support lemma we have $\bigcap S_\Delta \vdash \beta$, and so $\beta \in \Gamma$ for all $\Gamma \in S_\Delta$. This means that no $\Gamma \in W^c$ includes the set $\Delta \cup \{\neg\beta\}$. By Lemma 5.17, this is only possible if $\Delta \cup \{\neg\beta\} \vdash \bot$. From this we have $\Delta \vdash \neg\neg\beta$, and since for declaratives we have the double negation axiom, also $\Delta \vdash \beta$. By Lemma 5.5 it follows that $\Delta \vdash \psi$, contrary to assumption. □

5.4 Completeness for special frame classes

Particular interpretations of modal logic require restricting one's attention to special classes of Kripke frames, characterized by constraints on the relation R.

---

[6] As an aside, notice that if we restrict our language and our axioms to classical formulas, this proof still works, yielding a completeness result for the logic of strict implication in a classical setting over all Kripke models. Interestingly, our axiomatization is quite different from the one of Veltman [20]. A closer look at the relation must be left for another occasion.

Three of the most important constraints are reflexivity (wRw for all w) transitivity (wRv & vRu implies wRu) and Euclideanness (wRv & wRu implies vRu). In this section I provide axioms for $\Rightarrow$ corresponding to these conditions, and show that augmenting our system for $\mathsf{Inq}^{\Rightarrow}$ with these axioms leads to a system which is sound and complete for the corresponding class of frames. The relevant axioms are the following ones, where $\varphi, \psi, \chi \in \mathcal{L}^{\Rightarrow}$ and $\alpha, \beta \in \mathcal{L}_!$.

- Reflexivity: $(\alpha \Rightarrow \beta) \to (\alpha \to \beta)$
- Transitivity: $(\varphi \Rightarrow \psi) \to (\chi \Rightarrow (\varphi \Rightarrow \psi))$
- Euclideanness: $\neg(\varphi \Rightarrow \psi) \to (\chi \Rightarrow \neg(\varphi \Rightarrow \psi))$

Theorem 5.22 *Adding one or more of the above axioms to our proof system results in a sound and complete system for the class of frames whose accessibility relation has the properties corresponding to those axioms.*

As usual, soundness is shown directly, verifying that all instances of the above axioms are valid over the corresponding class. For the completeness result, we construct a canonical model M^c_+ as above, but based on the extended proof system $\vdash_+$. Proceeding exactly as above we can show that if $\Phi \nvdash_+ \psi$ then there is a state S in M^c_+ which supports all formulas in Φ but does not support ψ. Given this, we just need to show that the frame underlying the canonical model belongs to the relevant class. This is the role of the following lemma.

Lemma 5.23 (Canonicity) *If the proof system $\vdash_+$ includes one of the above axioms, then the canonical accessibility relation has the corresponding property.*

Proof.

- Suppose $\vdash_+$ contains all instances of the axiom for reflexivity. Consider a point Γ in the canonical model M^c_+. We want to show that $\Gamma R^c_+ \Gamma$. For this, take $\alpha, \beta \in \mathcal{L}_!$ and suppose $(\alpha \Rightarrow \beta) \in \Gamma$ and $\alpha \in \Gamma$: we need to show that $\beta \in \Gamma$. Since $(\alpha \Rightarrow \beta) \to (\alpha \to \beta)$ is an axiom, and since Γ contains $\alpha \Rightarrow \beta$ and is closed under deduction of declaratives in $\vdash_+$, we have $(\alpha \to \beta) \in \Gamma$. Since $\alpha \in \Gamma$, again by closure under deduction we have $\beta \in \Gamma$.
- Suppose $\vdash_+$ contains all instances of the axiom for transitivity. Suppose $\Gamma R^c_+ \Gamma'$ and $\Gamma' R^c_+ \Gamma''$. We want to show that $\Gamma R^c_+ \Gamma''$. For this, let $\alpha, \beta \in \mathcal{L}_!$ and suppose $(\alpha \Rightarrow \beta) \in \Gamma$ and $\alpha \in \Gamma''$: we need to show that $\beta \in \Gamma''$. Since $(\alpha \Rightarrow \beta) \to (\top \Rightarrow (\alpha \Rightarrow \beta))$ is an axiom and Γ is closed under deduction of declaratives in $\vdash_+$, we have $(\top \Rightarrow (\alpha \Rightarrow \beta)) \in \Gamma$. Since $\top \in \Gamma'$ and $\Gamma R^c_+ \Gamma'$, by definition of R^c_+ we have $(\alpha \Rightarrow \beta) \in \Gamma'$. Since $\alpha \in \Gamma''$ and $\Gamma' R^c_+ \Gamma''$, again by definition of R^c_+ we have $\beta \in \Gamma''$, as we wanted.
- Suppose $\vdash_+$ contains all instances of the axiom for Euclideanness. Suppose $\Gamma R^c_+ \Gamma'$ and $\Gamma R^c_+ \Gamma''$. We want to show that $\Gamma' R^c_+ \Gamma''$. For this, let $\alpha, \beta \in \mathcal{L}_!$ and suppose $(\alpha \Rightarrow \beta) \in \Gamma'$ and $\alpha \in \Gamma''$: we need to show that $\beta \in \Gamma''$. Towards a contradiction, suppose $\beta \notin \Gamma''$. Since $\Gamma R^c_+ \Gamma''$ and $\alpha \in \Gamma''$, by definition of R^c_+ it follows that $(\alpha \Rightarrow \beta) \notin \Gamma$. Since Γ is a complete theory of declaratives, $\neg(\alpha \Rightarrow \beta) \in \Gamma$. Since $\neg(\alpha \Rightarrow \beta) \to (\top \Rightarrow \neg(\alpha \Rightarrow \beta))$ is an axiom and Γ is closed under deduction of declaratives in $\vdash_+$, we have

$(\top \Rightarrow \neg(\alpha \Rightarrow \beta)) \in \Gamma$. Since $\top \in \Gamma'$ and $\Gamma R^c_+ \Gamma'$ we have $\neg(\alpha \Rightarrow \beta) \in \Gamma'$. But this is impossible, since Γ' is consistent and $(\alpha \Rightarrow \beta) \in \Gamma'$. □

6 Comparison with Goranko and Kuusisto's approach

In a recent paper [10], Goranko and Kuusisto (henceforth G&K) develop a logic of propositional dependency which shares some features of the present proposal. Indeed, the motivations for their work are similar to the ones that led us to investigate the system $\mathsf{Inq}^{\Rightarrow}$: e.g., G&K also aim at a system in which we can make good sense of negated and disjunctive dependence statements.

G&K's system is based on a modal language built up from atoms, connectives, and an operator D of flexible arity. Formulas are evaluated relative to a model $M = \langle W, V \rangle$ of the kind assumed above for InqB, and a world $w \in W$. Atoms and connectives are interpreted as usual; the clause for D is as follows:

$$M, w \models \mathsf{D}(\alpha_1, \dots, \alpha_n, \beta) \;\text{ iff }\; \exists f : \{0,1\}^n \to \{0,1\} \text{ s.t. } \forall v \in W : V(v, \beta) = f(V(v, \alpha_1), \dots, V(v, \alpha_n))$$

Now, with any model $M = \langle W, V \rangle$ we can associate a corresponding Kripke model $M^U = \langle W, U, V \rangle$, where U is the universal accessibility map, $U = W \times W$. Then, it is easy to see that we have the following equivalence.

$$M, w \models \mathsf{D}(\alpha_1, \dots, \alpha_n, \beta) \;\text{ iff }\; M^U, w \models ?\alpha_1 \wedge \dots \wedge ?\alpha_n \Rightarrow ?\beta$$

This connection allows us to bring out precisely what the differences are between the proposal of G&K and the one I described in this paper.

Firstly, G&K interpret dependency by means of a universal accessibility relation, while I allow it to be interpreted by means of an arbitrary relation. For some applications, this is necessary: e.g., in a multi-agent epistemic setting, different agents will have access to different information, and so different epistemic dependencies will hold from the perspective of each agent (see footnote 4). Modeling such scenarios requires using non-universal accessibility relations.

However, the main difference between the two approaches lies in the way the dependency relation is construed. In G&K's system, it is construed as a relation between statements, while in the view I propose it is construed as a relation between questions. An advantage of G&K's approach is that the semantics of the system can be kept completely truth-conditional. No detour at the level of information states is necessary. This contrasts with the situation in $\mathsf{Inq}^{\Rightarrow}$: although a dependence statement $\varphi \Rightarrow \psi$ itself is truth-conditional, its truth-conditions depend crucially on the support conditions for φ and ψ.

On the other hand, the account of dependence statements provided by $\mathsf{Inq}^{\Rightarrow}$ is more general than the one given by G&K. This is because it can deal with dependencies between questions which are not necessarily *polar* questions of the form $?\alpha$ for some statement α. For a concrete example, consider a conditional question like *whether Sue will dance if Bob asks her*. This question is resolved in a state s if all the worlds in s in which Bob asks Sue to dance agree on whether she accepts. This is captured by formalizing this question as $a \to ?d$.

$$M, s \models a \to ?d \;\text{ iff }\; M, s \cap |a|_M \models ?d$$

Notice that the conditional question does *not* correspond to the polar question $?(a \to d)$: establishing that Sue will not dance if Bob asks her ($a \to \neg d$) is sufficient to resolve the conditional question, but it does not determine whether the conditional $a \to d$ is true or false, and thus it does not resolve $?(a \to d)$.

Now consider the following dependence statement, where the determined question is the conditional question we just examined.

(2) Whether Sue is in a good mood determines whether she will dance if Bob asks her.

This does not amount to a dependency between the truth-value of two statements; thus, it cannot be formalized by G&K's D operator. By contrast, in our setting (2) can be expressed straightforwardly, following our general strategy, as a strict conditional $?g \Rightarrow (a \to ?d)$ involving the polar question $?g$ (*whether Sue is in a good mood*) and the conditional question $a \to ?d$ (*whether she will dance if Bob asks her*). Spelling out the clauses, for this formula we have:

$$M, w \models ?g \Rightarrow (a \to ?d) \quad \text{iff} \quad \exists f : \{0,1\} \to \{0,1\} \text{ such that } \forall v \in R[w] \cap |a|_M : V(v,d) = f(V(v,g))$$

Thus, (2) expresses the existence of a functional dependence of d on g not relative to the whole set $R[w]$ of accessible worlds, but only relative to a subset thereof, namely, the set of those accessible worlds where Bob asks Sue to dance. The system $\mathsf{Inq}^{\Rightarrow}$ gives us the means to express such more intricate dependence patterns, and to do so in a perspicuous, systematic, and compositional way. [7]

A further benefit of $\mathsf{Inq}^{\Rightarrow}$ is that it allows for a more fine-grained analysis of the logical operations involved in a dependence statement, which yields a better proof theory. Indeed, G&K's operator D translates to $\mathsf{Inq}^{\Rightarrow}$ as a rather complex combination of operators. Spelling out the abbreviation '?', the formula $\mathsf{D}(\alpha_1, \dots, \alpha_n, \beta)$ corresponds to our $((\alpha_1 \vee\!\!\!\vee \neg\alpha_1) \wedge \cdots \wedge (\alpha_n \vee\!\!\!\vee \neg\alpha_n)) \Rightarrow (\beta \vee\!\!\!\vee \neg\beta)$. This shows that D bears the burden of performing many operations at once: negation, inquisitive disjunction, conjunction, and the strict conditional. As a consequence, this operator is not simple from a proof-theoretic point of view. For instance, G&K's axiomatization of it contains the following axiom:

$$\mathsf{D}(\alpha_1, \dots, \alpha_n, \beta) \longleftrightarrow \bigvee_{\chi \in \mathrm{DNF}(\alpha_1, \dots, \alpha_n)} ((\chi \leftrightarrow \beta) \wedge \mathsf{D}(\chi \leftrightarrow \beta))$$

where $\mathrm{DNF}(\alpha_1, \dots, \alpha_n)$ is the set of disjunctions of formulas $\delta_1 \wedge \cdots \wedge \delta_n$ where $\delta_i \in \{\alpha_i, \neg\alpha_i\}$. Such an axiom is hardly the sort of characterization one would

[7] Indeed, notice that the translation of (2) as $?g \Rightarrow (a \to ?d)$ can be obtained systematically by applying the following translation rules, where $\rightsquigarrow$ abbreviates "translates to":

- if 'A' $\rightsquigarrow \alpha$ then 'whether A' $\rightsquigarrow ?\alpha$
- if 'A' $\rightsquigarrow \varphi$ and 'B' $\rightsquigarrow \psi$, then 'B if A' $\rightsquigarrow (\varphi \to \psi)$
- if 'A' $\rightsquigarrow \varphi$ and 'B' $\rightsquigarrow \psi$, then 'A determines B' $\rightsquigarrow (\varphi \Rightarrow \psi)$

Assuming that the syntactic parsing of (2) is "(whether (Sue is in a good mood)) determines ((whether (she will dance)) if (Bob asks her))", these rules deliver precisely $?g \Rightarrow (a \to ?d)$.

hope to have for the behavior of a primitive logical constant. By contrast, in $\mathsf{Inq}^{\Rightarrow}$ dependence statements are analyzed in terms of basic building blocks ($\bot, \to, \vee\!\!\!\vee, \wedge$, and $\Rightarrow$) each of which obeys simple and natural logical principles.

In spite of these differences in implementation, however, my proposal and the one of G&K converge on a fundamental conceptual point: dependence statements are modal statements, which are true or false at a world w according to whether a dependency holds in a set of worlds associated with w.

References

[1] Ciardelli, I., *Modalities in the realm of questions: axiomatizing inquisitive epistemic logic*, in: R. Goré, B. Kooi and A. Kurucz, editors, *Advances in Modal Logic* (2014), pp. 94–113.

[2] Ciardelli, I., *Dependency as question entailment*, in: S. Abramsky, J. Kontinen, J. Väänänen and H. Vollmer, editors, *Dependence Logic: theory and applications*, Springer International Publishing Switzerland, 2016 pp. 129–181.

[3] Ciardelli, I., "Questions in logic," Ph.D. thesis, ILLC, University of Amsterdam (2016).

[4] Ciardelli, I., *Questions as information types*, Synthese **195** (2018), pp. 321–365.

[5] Ciardelli, I., J. Groenendijk and F. Roelofsen, *Inquisitive semantics: a new notion of meaning*, Language and Linguistics Compass **7** (2013), pp. 459–476.

[6] Ciardelli, I., R. Iemhoff and F. Yang, *Questions and dependency in intuitionistic logic* (2018), manuscript, under review.

[7] Ciardelli, I. and F. Roelofsen, *Inquisitive logic*, Journal of Philosophical Logic **40** (2011), pp. 55–94.

[8] Ciardelli, I. and F. Roelofsen, *Inquisitive dynamic epistemic logic*, Synthese **192** (2015), pp. 1643–1687.

[9] Ebbing, J., L. Hella, A. Meier, J.-S. Müller, J. Virtema and H. Vollmer, *Extended modal dependence logic*, in: L. Libkin, U. Köhlenbach and R. de Queiroz, editors, *Proceedings of WoLLIC 2013*, Lecture Notes in Computer Science (2013), pp. 126–137.

[10] Goranko, V. and A. Kuusisto, *Logics for propositional determinacy and independence*, Review of Symbolic Logic (2018), DOI: 10.1007/s11229-016-1221-y.

[11] Groenendijk, J., *The logic of interrogation*, in: T. Matthews and D. Strolovitch, editors, *Semantics and Linguistic Theory* (1999), pp. 109–126.

[12] Groenendijk, J., *Inquisitive semantics: Two possibilities for disjunction*, in: P. Bosch, D. Gabelaia and J. Lang, editors, *Tbilisi Symposium on Language, Logic, and Computation*, Springer-Verlag, 2009 pp. 80–94.

[13] Hintikka, J., "Knowledge and Belief: an Introduction to the Logic of the Two Notions," Cornell University Press, 1962.

[14] Mascarenhas, S., *Inquisitive semantics and logic* (2009), Master Thesis, University of Amsterdam.

[15] Puncochar, V., *Weak negation in inquisitive semantics*, Journal of Logic, Language, and Information **24** (2015), pp. 323–355.

[16] Puncochar, V., *A generalization of inquisitive semantics*, Journal of Philosophical Logic **45** (2016), pp. 399–428.

[17] Puncochar, V., *Substructural inquisitive logics* (2018), manuscript, under review.

[18] Väänänen, J., "Dependence Logic: A New Approach to Independence Friendly Logic," Cambridge University Press, Cambridge, 2007.

[19] Väänänen, J., *Modal dependence logic.*, in: K. Apt and R. van Rooij, editors, *New Perspectives on Games and Interaction*, Amsterdam Univ. Press, 2008 pp. 237–254.

[20] Veltman, F., "Logics for Conditionals," Ph.D. thesis, University of Amsterdam (1985).

[21] Yang, F. and J. Väänänen, *Propositional logics of dependence*, Annals of Pure and Applied Logic **167** (2016), pp. 557–589.