An Ehrenfeucht–Fraïssé game for inquisitive first-order logic*

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Abstract. Inquisitive first-order logic, InqBQ, is an extension of classical first-order logic with questions. From a mathematical point of view, formulas in this logic express properties of sets of relational structures. In this paper we describe an Ehrenfeucht-Fraïssé game for InqBQ and show that it characterizes the distinguishing power of the logic. We exploit this result to show a number of undefinability results: in particular, several variants of the question *how many individuals have property* P are not expressible in InqBQ, even in restriction to finite models.

1 Introduction

According to the traditional view, the semantics of a logical system specifies truth-conditions for the sentences in the language. This focus on truth restricts the scope of logic to a special kind of sentences, namely, statements, whose semantics can be adequately characterized in terms of truth-conditions. In recent years, a more general view of semantics has been developed, which goes under the name of *inquisitive* semantics (see [4] for a language-oriented introduction, and [2] for a logic-oriented one). In this approach, the meaning of a sentence is laid out not by specifying when the sentence is true relative to a state of affairs, but rather by specifying when it is supported by a given state of information. This view allows us to interpret in a uniform way both statements and questions: for instance, the statement *it rains* will be supported by an information state s if the information available in s implies that it rains, while the question *whether it rains* will be supported by s if the information available in s determines whether or not it rains.

In its first-order version, the system InqBQ, inquisitive logic can be seen as a conservative extension of classical first-order logic with formulas expressing questions. Thus, in addition to standard first-order formulas like Pa and $\forall x Px$, we also have formulas like $?Pa$ (*whether* a *has property* P), $\overline{\exists} x Px$ (*what is an instance of property* P), and $\forall x ?Px$ (*which individuals have property* P). A model for this logic is based on a set W of possible worlds, each representing

* Financial support by the European Research Council under the EU's Horizon 2020 research and innovation programme (grant No. 680220) is gratefully acknowledged.

a possible state of affairs, corresponding to a standard first-order structure. An information state is modeled as a subset $s \subseteq W$. The idea, which goes back to the work of Hintikka [13], is that a set of worlds s stands for a body of information that is compatible with the actual world being one of the worlds $w \in s$, and incompatible with it being one of the worlds $w \notin s$. The semantics of the language takes the form of a support relation holding between information states in a model and sentences of the language.[3]

From a mathematical point of view, a sentence of InqBQ expresses a property of a set s of first-order structures. The crucial difference between statements and questions is that statements express *local* properties of information states—which boil down to requirements on the individual worlds $w \in s$—while questions express *global* requirements, having to do with the way the worlds in s are related to each other. Thus, for instance, the formula $?Pa$ requires that the truth-value of Pa be the same in all worlds in s; the formula $\overline{\exists} xPx$ requires that there be an individual that has property P uniformly in all worlds in s; and the formula $\forall x?Px$ requires that the extension of property P be the same in across s. Global properties can also take the form of *dependencies*: thus, e.g., $?Pa \to ?Qa$ requires that the truth-value of Qa be functionally determined by the truth-value of Pa in s, while $\forall x?Px \to \forall x?Qx$ requires that the extension of property Q be functionally determined by the extension of property P in s. Thus, inquisitive first-order logic provides a natural language to talk about both local and global features of an information state.

In contrast to propositional inquisitive logic, which has been thoroughly investigated (see, among others, [1,5,21,19,20,9,3]), first-order inquisitive logic has received comparatively little attention [2,12]. In particular, a detailed investigation of the expressive power of the logic has so far been missing. This paper is a first step in this direction.

In the classical context, a powerful tool to study the expressiveness of first-order logic is provided by Ehrenfeucht-Fraïssé games (also known as EF games or back-and-forth games), introduced in 1967 by Ehrenfeucht [6], developing model-theoretic results presented by Fraïssé [8]. These games provide a particularly perspicuous way of understanding what differences between models can be detected by means of first-order formulas of a certain quantifier rank. Reasoning about winning strategies in this game, one can prove that two first-order structures are elementarily equivalent, or one can find a formula telling them apart. As an application, EF games provide relatively easy proofs that certain properties of first-order structures are not first-order expressible.

The basic idea of EF games has proven to be extremely flexible and adaptable to a wide range of logical settings, including fragments of first-order logic with finitely many variables [16]; extensions of first-order logic with generalized quantifiers [17]; monadic second order logic [7]; modal logic [24]; and intuitionistic logic [25,18]. In each case, the game provides an insightful characterization

[3] Thus, our logic fits within the family of logic based on *team semantics*, along with, e.g., dependence and independence logic ([14,22,10,11,26], among others).

of the distinctions that can and cannot be made by means of formulas in the logic.

Our aim in this paper is to describe an EF-style game for the logic InqBQ and to show that this game provides a characterization of the expressive power of InqBQ. As an application, we show that certain natural questions are not expressible in InqBQ: in particular, the question *how many individuals have property* P (supported in s if the extension of P has the same cardinality in all the worlds in s) and the question *whether there are only finitely many individuals satisfying* P (supported in s if the extension of P is finite in all the worlds in s, or infinite in all the worlds in s).

The paper is structured as follows: in Section 2 we provide some technical background on the logic InqBQ. In Section 3 we describe the game and prove our main result, linking winning strategies for bounded versions of the game to the distinguishing power of fragments of InqBQ. In Section 4 we use this result to show that certain questions are not expressible in InqBQ. In Section 5 we summarize our findings and mention some directions for future work.

2 Inquisitive first-order logic

In this section we provide a basic introduction to inquisitive first-order logic. For a more comprehensive introduction, the reader is referred to [2].

Let Σ be a predicate logic signature. For simplicity, we first restrict to the case in which Σ is a *relational* signature, i.e., contains no function symbols. The extension to an arbitrary signature, which involves some subtleties familiar from the classical case [15], is discussed in Section 3.4. The set $\mathcal{L}$ of formulas of InqBQ over Σ is defined as follows, where $R \in \Sigma$ is an n-ary relation symbol:

$$\varphi ::= R(x_1, \ldots, x_n) \mid (x_1 = x_2) \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \forall x \varphi \mid \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \varphi \mid \overline{\exists} x \varphi$$

Formulas without occurrences of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\overline{\exists}$ are referred to as *classical* formulas, and can be identified with standard first-order logic formulas. If α and β are classical formulas, then we can define $\neg\alpha := \alpha \to \bot$, $\alpha \vee \beta := \neg(\neg\alpha \wedge \neg\beta)$ and $\exists x \alpha := \neg\forall x \neg\alpha$. The set of classical formulas is denoted $\mathcal{L}_c$.

The connective $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and the quantifier $\overline{\exists}$, referred to respectively as *inquisitive disjunction* and *inquisitive existential quantifier*, allow us to form questions. For instance, if α is a classical formula then the formula $?\alpha := \alpha \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg\alpha$ represents the question *whether* α; the formula $\overline{\exists} x \alpha(x)$ represents the question *what is an individual satisfying* $\alpha(x)$; and the formula $\forall x ?\alpha(x)$ represents the question *which individuals satisfy* $\alpha(x)$.

A model for InqBQ consists of a set W of worlds—representing possible states of affairs—a set D of individuals, and an interpretation function I which determines at each world the extension of all relation symbols, including identity.

Definition 1 (Models).
A model for the signature Σ *is a tuple* $M = \langle W, D, I \rangle$ *where* W *and* D *are sets and* I *is a function mapping each world* $w \in W$ *and each* n*-ary relation symbol*

$R \in \Sigma \cup \{=\}$ to a corresponding n-ary relation $I_w(R) \subseteq D^n$—the extension of R at w. The interpretation of identity is subject to the following condition:

> *$I_w(=)$ is an equivalence relation $\sim_w$ which is a congruence, i.e., if $R \in \Sigma$ and $d_i \sim_w d'_i$ for $i \leq n$, then $\langle d_1, \ldots, d_n \rangle \in I_w(R) \iff \langle d'_1, \ldots, d'_n \rangle \in I_w(R)$.*

As discussed in the introduction, in inquisitive logic the semantics of the language specifies when a formula is supported at an information state $s \subseteq W$, rather than when a formula is true at a possible world $w \in W$. As usual, to handle open formulas and quantification, the support relation is defined relative to an assignment, which is a function from variables to the set D of individuals; if g is an assignment and $d \in D$, then $g[x \mapsto d]$ is the assignment which maps x to d and behaves like g on all other variables.

Definition 2 (Support). *Let $M = \langle W, D, I \rangle$ be a model and let $s \subseteq W$.*

$$
\begin{array}{lll}
M, s \models_g R(x_1, \ldots, x_n) & \iff & \forall w \in s : \langle g(x_1), \ldots, g(x_n) \rangle \in I_w(R) \\
M, s \models_g x_1 = x_2 & \iff & \forall w \in s : g(x_1) \sim_w g(x_2) \\
M, s \models_g \bot & \iff & s = \emptyset \\
M, s \models_g \varphi \wedge \psi & \iff & M, s \models_g \varphi \text{ and } M, s \models_g \psi \\
M, s \models_g \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geq$}} \psi & \iff & M, s \models_g \varphi \text{ or } M, s \models_g \psi \\
M, s \models_g \varphi \to \psi & \iff & \forall t \subseteq s : M, t \models_g \varphi \text{ implies } M, t \models_g \psi \\
M, s \models_g \forall x \varphi & \iff & M, s \models_{g[x \mapsto d]} \varphi \text{ for all } d \in D \\
M, s \models_g \exists x \varphi & \iff & M, s \models_{g[x \mapsto d]} \varphi \text{ for some } d \in D
\end{array}
$$

As usual, if $\varphi(x_1, \ldots, x_n)$ is a formula whose free variables are among $x_1, \ldots, x_n$, then the value of g on variables other than $x_1, \ldots, x_n$ is irrelevant. If $d_1, \ldots, d_n \in D$, we can therefore write $M, s \models \varphi(d_1, \ldots, d_n)$ to mean that $M, s \models_g \varphi$ holds with respect to an assignment g that maps x_i to d_i. In particular, if φ is a sentence we can drop reference to g altogether. Moreover, we write $M \models \varphi$ as a shorthand for $M, W \models \varphi$ and we say that M supports φ.

The support relation has the following two basic features:

- Persistency: if $M, s \models_g \varphi$ and $t \subseteq s$ then $M, t \models_g \varphi$;
- Empty state property: if $M, \emptyset \models_g \varphi$ for all φ.

In restriction to classical formulas, the above definition of support gives a non-standard semantics for classical first-order logic. To see why, let us associate to each world $w \in M$ a corresponding relational structure $\mathcal{M}_w$, having as its domain the quotient $D/\sim_w$, and with the interpretation of relation symbols induced by $I_w(R)$. Then for all classical formulas $\alpha \in \mathcal{L}_c$ we have:

$$M, s \models_g \alpha \iff \forall w \in s : \mathcal{M}_w \models_{\overline{g}} \alpha \text{ holds in first-order logic}$$

where $\overline{g}$ is the assignment mapping x to the $\sim_w$-equivalence class of $g(x)$. Thus, as far as the standard fragment of the language is concerned, the relation of support is essentially a recursive definition of global truth with respect to a set of structures sharing the same domain. Notice that the standard definition of truth can be recovered as a special case of support by taking s to be a singleton. We will also write $M, w \models_g \alpha$ as an abbreviation for $M, \{w\} \models_g \alpha$.

Thus, evaluating a classical formula on an information state s amounts to evaluating it at each world in s and determining whether it is satisfied at each world. The same is not true for formulas that contain the operators $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\overline{\exists}$; typically, such formulas allow us to express global requirements on a state, which cannot be reduced to requirements on the single worlds in the state. We will illustrate this point by means of some examples. First take a classical sentence α, and consider the formula $?\alpha := \alpha \vee\!\!\!\vee \neg\alpha$. We have:

$$M, s \models ?\alpha \iff (\forall w \in s : M, w \models \alpha) \text{ or } (\forall w \in s : M, w \models \neg\alpha)$$

Thus, in order for s to support $?\alpha$, all the worlds in s must agree on the truth-value of α. In other words, $?\alpha$ is supported at s only if the information available in s determines whether or not α is true.

Next take $\alpha(x)$ to be a classical formula having only the variable x free, and consider the formula $\overline{\exists}x\alpha(x)$. We have:

$$M, s \models \overline{\exists}x\alpha(x) \iff \exists d \in D \text{ s.t. } \forall w \in s : M, w \models \alpha(d)$$

Thus, in order for s to support $\overline{\exists}x\alpha(x)$ there must be an individual d which satisfies $\alpha(x)$ at all worlds in s. In other words, $\overline{\exists}x\alpha(x)$ is supported at s if the information available in s implies for some specific individual that it satisfies $\alpha(x)$—i.e., gives us a specific witness for $\alpha(x)$.

Finally, let again $\alpha(x)$ be a classical formula having only x free, and let us denote by α_w the extension of $\alpha(x)$ at w, i.e., $\alpha_w := \{d \in D \mid M, w \models \alpha(d)\}$. Consider the formula $\forall x?\alpha(x) := \forall x(\alpha(x) \vee\!\!\!\vee \neg\alpha(x))$. We have:

$$M, s \models \forall x\alpha(x) \iff \forall w, w' \in s : \alpha_w = \alpha_{w'}$$

Thus, in order for s to support $\forall x?\alpha(x)$, all the worlds in s must agree on the extension that they assign to $\alpha(x)$. In other words, $\forall x?\alpha(x)$ is supported at s if the information available in s determines exactly which individuals satisfy $\alpha(x)$.

An aspect of InqBQ which is worth commenting on is the interpretation of identity. In InqBQ, the interpretation of identity may differ at different worlds. This allows to deal with uncertainty about the identity relation: e.g., one may have information about two individuals, a and b (say, one knows Pa and Qb) and yet be uncertain whether a and b are distinct individuals, or the same. This also allows for uncertainty about how many individuals there are. Indeed, although a model is based on a fixed set D of epistemic individuals—objects to which information can be attributed—the domain of actual individuals at a world w is given by the equivalence classes modulo $\sim_w$; the number of actual individuals that exist at w is the number of such equivalence classes, i.e., the cardinality of the quotient $D/{\sim_w}$. Of course, the case in which the identity is treated as rigid can be recovered by taking $\sim_w$ to be the actual relation of identity on D at each world. A model in which identity is treated in this way is called an id-model.

This section is only intended as a summary of the relevant notions and as a quick illustration of the features of information states that can be expressed by means of formulas of InqBQ. With these basic notions in place and hopefully some grasp of how the logic works, we are now ready to turn to the novel contribution of the paper: an Ehrenfeucht-Fraïssé game for InqBQ.

3 An Ehrenfeucht-Fraïssé game for InqBQ

The EF game for InqBQ is played by two players, S (Spoiler) and D (Duplicator), using two inquisitive models M_0, M_1 as a board. As in the classical case, the game proceeds in turns: at each turn, S picks an object from one of the two models and D must respond by picking a corresponding object from the other model. At the end of the game, a winner is decided by comparing the atomic formulae supported by the sub-structures built during the game.

However, there are two crucial differences with the classical EF game. First, the objects that are picked during the game are not just individuals $d \in D_i$, but also information states $s \subseteq W_i$. This is because the logical repertoire of InqBQ contains not only the operators $\forall$ and $\overline{\exists}$, which quantify over individuals, but also the operator $\rightarrow$, which quantifies over information states. Second, the roles of the two models in the game are not symmetric. This is connected to the absence of a classical negation in the language of InqBQ; unlike in classical logic, it could be that a model M_0 supports all the formulas supported by a model M_1, but not viceversa. This directionality is reflected by the game.

3.1 The game

A position in an EF game for InqBQ is a tuple $\langle M_0, s_0, \overline{a}_0; M_1, s_1, \overline{a}_1 \rangle$ where:

- $M_0 = \langle W_0, D_0, I_0 \rangle$ and $M_1 = \langle W_1, D_1, I_1 \rangle$ are models for InqBQ;
- s_0 and s_1 are information states in the models M_0 and M_1 respectively;
- $\overline{a}_0$ and $\overline{a}_1$ are tuples of equal size of elements from D_0 and D_1 respectively.

Starting a round from a position $\langle M_0, s_0, \overline{a}_0; M_1, s_1, \overline{a}_1 \rangle$, the following are the possible moves:[4]

- $\overline{\exists}$-move: S picks an element $b_0 \in D_0$; D responds with an element $b_1 \in D_1$; the game continues from the position $\langle M_0, s_0, \overline{a}_0 b_0; M_1, s_1, \overline{a}_1 b_1 \rangle$;
- $\forall$-move: S picks an element $b_1 \in D_1$; D responds with an element $b_0 \in D_0$; the game continues from the position $\langle M_0, s_0, \overline{a}_0 b_0; M_1, s_1, \overline{a}_1 b_1 \rangle$;
- $\rightarrow$-move: S picks a sub-state $t_1 \subseteq s_1$; D responds with a sub-state $t_0 \subseteq s_0$; S picks $i \in \{1, 0\}$. The game continues from $\langle M_i, t_i, \overline{a}_i; M_{1-i}, t_{1-i}, \overline{a}_{1-i} \rangle$.

Notice the asymmetry between the roles of the two models: by performing an $\rightarrow$-move, S can pick an information state from M_1, but not a state in M_0.

With respect to termination condition, we consider different versions of the game. In the bounded version of the game, a pair of numbers $\langle i, q \rangle \in \mathbb{N}^2$ is fixed in advance. This number constrains the development of the game: in total, S can play only i implication moves and only q quantifier moves (i.e., $\overline{\exists}$-move or a $\forall$-move). When there are no more moves available, the game ends. If $\langle M_0, s_0, \overline{a}_0; M_1, s_1, \overline{a}_1 \rangle$ is the final position, the game is won by Player D if the following condition is satisfied, and by player S otherwise:

[4] In the following, the notation $\overline{a}b$ indicates the sequence obtained my adding the element b at the end of the sequence $\overline{a}$.

– Winning condition for D: for all atomic formulas $\alpha(x_1, \dots, x_n)$ where n is the size of the tuples $\overline{a}_0$ and $\overline{a}_1$, we have:

$$M_0, s_0 \models \alpha(\overline{a}_0) \implies M_1, s_1 \models \alpha(\overline{a}_1) \quad (1)$$

In the unbounded version of the game, no restriction is placed at the outset on the number of moves to be performed. Instead, player S has the option to declare the game over at the beginning of each round: in this case, the winner is determined as in the bounded version of the game. If the game never stops, then D is the winner.[5]

If D has a winning strategy in the EF game of lenght $\langle i, q\rangle$ starting from the position $\langle M_0, s_0, \overline{a}_0; M_1, s_1, \overline{a}_1\rangle$ we write:

$$M_0, s_0, \overline{a}_0 \preceq_{i,q} M_1, s_1, \overline{a}_1$$

We write $\approx_{i,q}$ for the relation $[\preceq_{i,q}] \cap [\succeq_{i,q}]$. Notice that, if $M_0, s_0, \overline{a}_0 \preceq_{i,q} M_1, s_1, \overline{a}_1$ does not hold, then it follows from the Gale-Stewart Theorem that S has a winning strategy in the EF game of length $\langle i, q\rangle$ starting from the position $\langle M_0, s_0, \overline{a}_0; M_1, s_1, \overline{a}_1\rangle$.

We use the notation $\mathcal{M}_0 \preceq_{i,q} \mathcal{M}_1$ as a shorthand for $\mathcal{M}_0, W^{\mathcal{M}_0}, \epsilon \preceq_{i,q} \mathcal{M}_1, W^{\mathcal{M}_1}, \epsilon$, where ϵ stands for the empty tuple.

The following Propositions follow easily from the definition of the game.

Proposition 1. *If $M_0, s_0, \overline{a}_0 \preceq_{i,q} M_1, s_1, \overline{a}_1$ then $M_0, s_0, \overline{a}_0 \preceq_{i',q'} M_1, s_1, \overline{a}_1$ for all $i' \leq i$ and $q' \leq q$.*

Proposition 2. *Suppose $\langle i, q\rangle \neq \langle 0, 0\rangle$. $M_0, s_0, \overline{a}_0 \preceq_{i,q} M_1, s_1, \overline{a}_1$ iff the following three conditions are satisfied:*

– *If $i > 0$, then $\forall t_1 \subseteq s_1 \ \exists t_0 \subseteq s_0 : M_0, t_0, \overline{a}_0 \approx_{i-1,q} M_1, t_1, \overline{a}_1$*
– *If $q > 0$, then $\forall b_0 \in D_0 \ \exists b_1 \in D_1 : M_0, s_0, \overline{a}_0 b_0 \preceq_{i,q-1} M_1, s_1, \overline{a}_1 b_1$*
– *If $q > 0$, then $\forall b_1 \in D_1 \ \exists b_0 \in D_0 : M_0, s_0, \overline{a}_0 b_0 \preceq_{i,q-1} M_1, s_1, \overline{a}_1 b_1$*

3.2 IQ degree and types

We define the *implication degree* (Ideg) and *quantification degree* (Qdeg) of a formula by the following inductive clauses, where p stands for an atomic formula:

$\mathrm{Ideg}(p)$	$= 0$	$\mathrm{Qdeg}(p)$	$= 0$
$\mathrm{Ideg}(\bot)$	$= 0$	$\mathrm{Qdeg}(\bot)$	$= 0$
$\mathrm{Ideg}(\varphi_1 \wedge \varphi_2)$	$= \max(\mathrm{Ideg}(\varphi_i))$	$\mathrm{Qdeg}(\varphi_1 \wedge \varphi_1)$	$= \max(\mathrm{Qdeg}(\varphi_i))$
$\mathrm{Ideg}(\varphi_1 \vee\!\!\!\vee \varphi_2)$	$= \max(\mathrm{Ideg}(\varphi_i))$	$\mathrm{Qdeg}(\varphi_1 \vee\!\!\!\vee \varphi_1)$	$= \max(\mathrm{Qdeg}(\varphi_i))$
$\mathrm{Ideg}(\varphi_1 \to \varphi_2)$	$= \max(\mathrm{Ideg}(\varphi_i)) + 1$	$\mathrm{Qdeg}(\varphi_1 \to \varphi_1)$	$= \max(\mathrm{Qdeg}(\varphi_i))$
$\mathrm{Ideg}(\forall x \varphi)$	$= \mathrm{Ideg}(\varphi)$	$\mathrm{Qdeg}(\forall x \varphi)$	$= \mathrm{Qdeg}(\varphi) + 1$
$\mathrm{Ideg}(\exists\!\!\!\exists x \varphi)$	$= \mathrm{Ideg}(\varphi)$	$\mathrm{Qdeg}(\exists\!\!\!\exists x \varphi)$	$= \mathrm{Qdeg}(\varphi) + 1$

[5] Here we consider games in which the rounds of play are indexed by natural numbers. To define games of transfinite length, one would have to specify how to determine the game position corresponding to a limit ordinal. We leave this for future work.

The combined degree of a formula is defined as $\mathrm{IQdeg}(\varphi) = \langle \mathrm{Ideg}(\varphi), \mathrm{Qdeg}(\varphi)\rangle$. We define an order relation $\leq$ on such degrees by setting:

$$\langle a, b\rangle \leq \langle a', b'\rangle \iff a \leq a' \text{ and } b \leq b'$$

We denote by $\mathcal{L}^l_{i,q}$ the set of formulas φ such that $\mathrm{IQdeg}(\varphi) \leq \langle i, q\rangle$ and the set of free variables in φ is included in $\{x_1, \ldots, x_l\}$. We can then define the key notion of $\langle i, q\rangle$-type.

Definition 3 ($\langle i, q\rangle$-types). *Let M be a model, s an information state, and $\overline{a}$ a tuple of elements in M of length l. The $\langle i, q\rangle$-type of $\langle M, s, \overline{a}\rangle$ is the set*

$$\mathrm{tp}_{i,q}(M, s, \overline{a}) := \left\{\varphi \in \mathcal{L}^l_{i,q} \;\middle|\; M, s \models \varphi(\overline{a})\right\}$$

We also define the following notation:

$$M_0, s_0, \overline{a}_0 \sqsubseteq_{i,q} M_1, s_1, \overline{a}_1 \overset{def}{\iff} \mathrm{tp}_{i,q}(M_0, s_0, \overline{a}_0) \subseteq \mathrm{tp}_{i,q}(M_1, s_1, \overline{a}_1)$$
$$M_0, s_0, \overline{a}_0 \equiv_{i,q} M_1, s_1, \overline{a}_1 \overset{def}{\iff} \mathrm{tp}_{i,q}(M_0, s_0, \overline{a}_0) = \mathrm{tp}_{i,q}(M_1, s_1, \overline{a}_1)$$

Notice that, if the signature is finite, there are only a finite number of non-equivalent formulas of combined degree at most $\langle i, q\rangle$, and consequently only a finite number of $\langle i, q\rangle$-types. This can be shown through a simple inductive argument:

- there are only finite (up to logical equivalence) formulas in $\mathcal{L}^l_{0,0}$—atoms and their boolean combinations. Notice the hypothesis of working with a finite signature is necessary here.
- formulas in $\mathcal{L}^l_{i,q}$ are equivalent to boolean combinations of formulas in $A \cup B$, for $A = \left\{\varphi \to \psi \;\middle|\; \varphi, \psi \in \mathcal{L}^l_{i-1,q}\right\}$ and $B = \left\{\overline{\exists} x.\varphi, \forall x.\varphi \;\middle|\; \varphi \in \mathcal{L}^{l+1}_{i,q-1}\right\}$—where we impose by definition $\mathcal{L}^l_{i,q} = \emptyset$ if $i < 0$ or $q < 0$. A and B contain only finitely many non-equivalent formulas, and so does $\mathcal{L}^l_{i,q}$.

3.3 The EF theorem

What follows is the main result of the paper: the relations $\preceq_{i,q}$ and $\sqsubseteq_{i,q}$ coincide.

Theorem 1. *Suppose the signature Σ is finite. Then*

$$M_0, s_0, \overline{a}_0 \preceq_{i,q} M_1, s_1, \overline{a}_1 \quad\iff\quad M_0, s_0, \overline{a}_0 \sqsubseteq_{i,q} M_1, s_1, \overline{a}_1$$

Proof. We will prove this by well-founded induction on $\langle i, q\rangle$. For the basic case, $\langle i, q\rangle = \langle 0, 0\rangle$, we just have to verify that, if Condition (1) holds for all atomic formulas, then holds for all formulas $\varphi \in \mathcal{L}^l_{0,0}$. This is straightforward. Next, suppose $\langle i, q\rangle > \langle 0, 0\rangle$ and suppose the claim holds for $\langle i', q'\rangle < \langle i, q\rangle$. For the left-to-right direction, we proceed by contraposition. Suppose that for some $\varphi \in \mathcal{L}^l_{i,q}$

$$M_0, s_0 \models \varphi(\overline{a}_0) \qquad M_1, s_1 \not\models \varphi(\overline{a}_1)$$

W.l.o.g., we can assume that φ is of the form $\psi \to \chi$, $\forall x\psi$ or $\overline{\exists} x\psi$. Indeed:

– If φ is an atom, it follows $M_0, s_0, \overline{a}_0 \not\preceq_{0,0} M_1, s_1, \overline{a}_1$ and thus by Proposition 1 also $M_0, s_0, \overline{a}_0 \not\preceq_{i,q} M_1, s_1, \overline{a}_1$. Thus, in this case the conclusion follows.
– If φ is a conjunction $\psi \wedge \chi$ then we have:

$$\begin{cases} M_0, s_0 \models \psi(\overline{a}_0) \wedge \chi(\overline{a}_0) \implies M_0, s_0 \models \psi(\overline{a}_0) \text{ and } M_0, s_0 \models \chi(\overline{a}_0) \\ M_1, s_1 \not\models \psi(\overline{a}_1) \wedge \chi(\overline{a}_1) \implies M_1, s_1 \not\models \psi(\overline{a}_1) \text{ or } M_1, s_1 \not\models \chi(\overline{a}_1) \end{cases}$$

So, either ψ or χ is a less complex witness of $M_0, s_0, \overline{a}_0 \not\sqsubseteq_{i,q} M_1, s_1, \overline{a}_1$.
– If φ is a disjunction $\psi \vee\!\!\!\vee \chi$, we can reach a conclusion analogous to the one we reached for conjunction.

Let us consider the three cases separately.

Case 1: φ is an implication $\psi \to \chi$. In this case we have

$$M_1, s_1 \not\models \psi(\overline{a}_1) \to \chi(\overline{a}_1) \implies \exists (t_1 \subseteq s_1)\, [M_1, t_1 \models \psi(\overline{a}_1) \text{ and } M_1, t_1 \not\models \chi(\overline{a}_1)]$$
$$M_0, s_0 \models \psi(\overline{a}_0) \to \chi(\overline{a}_0) \implies \nexists (t_0 \subseteq s_0)\, [M_0, t_0 \models \psi(\overline{a}_0) \text{ and } M_0, t_0 \not\models \chi(\overline{a}_0)]$$

Thus there exists a state $t_1 \subseteq s_1$ with a different $\langle i-1, q\rangle$-type than every $t_0 \subseteq s_0$—either because it supports ψ or because it does not support χ. So by induction hypothesis, if S performs a $\to$-move and chooses t_1, for every choice t_0 of D we have $M_0, t_0, \overline{a}_0 \not\approx_{i-1,q} M_1, t_1, \overline{a}_1$. It follows by Proposition 2 that $M_0, s_0, \overline{a}_0 \not\preceq_{i,q} M_1, s_1, \overline{a}_1$ as wanted.

Case 2: φ is a universal $\forall x \psi$. In this case we have

$$M_1, s_1 \not\models \forall x \psi(\overline{a}_1, x) \implies \exists (b_1 \in D_1)\; M_1, s_1 \not\models \psi(\overline{a}_1, b_1)$$
$$M_0, s_0 \models \forall x \psi(\overline{a}_0, x) \implies \forall (b_0 \in D_0)\; M_0, s_0 \models \psi(\overline{a}_0, b_0)$$

Thus if S performs a $\forall$-move and chooses b_1, for every choice b_0 of D, by induction hypothesis we have

$$M_0, s_0, \overline{a}_0 b_0 \;\not\preceq_{i,q-1} M_1, s_1, \overline{a}_1 b_1$$

It follows by Proposition 2 that $M_0, s_0, \overline{a}_0 \not\preceq_{i,q} M_1, s_1, \overline{a}_1$ as wanted.

Case 3: φ is an inquisitive existential $\overline{\exists} x \psi$. This case is similar to the previous one: S can perform an $\overline{\exists}$-move and pick an element b_0 in D_0 with no counterpart in D_1, and by Proposition 2 we get the result.

This completes the proof of the left-to-right direction of the inductive step. Now consider the converse direction. Again, we proceed by contraposition. Suppose that S has a winning strategy in the EF game of length $\langle i, q\rangle$ starting from $(M_0, s_0, \overline{a}_0; M_1, s_1, \overline{a}_1)$. We consider again three cases, depending on the first move of the winning strategy.

Case 1: the first move is a $\to$-move. Suppose S starts by choosing $t_1 \subseteq s_1$. As this is a winning strategy for S, for every choice $t_0 \subseteq s_0$ of D we have

$$M_0, t_0, \overline{a}_0 \not\preceq_{i-1,q} M_1, t_1, \overline{a}_1 \quad \text{or} \quad M_1, t_1, \overline{a}_1 \not\preceq_{i-1,q} M_0, t_0, \overline{a}_0$$

By inductive hypothesis, this translates to

$$\exists \psi_{t_0} \in \text{tp}(t_0) \setminus \text{tp}(t_1) \quad \text{or} \quad \exists \theta_{t_0} \in \text{tp}(t_1) \setminus \text{tp}(t_0)$$

where $\mathrm{tp}(t_0) := \mathrm{tp}_{i-1,q}(M_0, t_0, \overline{a}_0)$ and $\mathrm{tp}(t_1) := \mathrm{tp}_{i-1,q}(M_1, t_1, \overline{a}_1)$.

Given this, there exist two families $\{\psi_{t_0} \mid t_0 \subseteq s_0\}$ and $\{\theta_{t_0} \mid t_0 \subseteq s_0\}$ s.t.:

$$\begin{cases} \psi_{t_0} \in \mathrm{tp}(t_0) \setminus \mathrm{tp}(t) & \text{if } \mathrm{tp}(t_0) \setminus \mathrm{tp}(t) \neq \emptyset \\ \psi_{t_0} := \bot & \text{otherwise} \end{cases}$$

$$\begin{cases} \theta_{t_0} \in \mathrm{tp}(t) \setminus \mathrm{tp}(t_0) & \text{if } \mathrm{tp}(t) \setminus \mathrm{tp}(t_0) \neq \emptyset \\ \theta_{t_0} := \top & \text{otherwise} \end{cases}$$

Moreover, we can suppose the two families to be finite, as there are only a finite number of formulas of degree $\langle i-1, q\rangle$ up to logical equivalence (see Subsection 3.2).

Define now

$$\varphi := \bigwedge_{t_0 \subseteq s_0} \theta_{t_0} \to \bigvee\!\!\!\!\bigvee_{t_0 \subseteq s_0} \psi_{t_0}$$

We have: (i) $\mathrm{IQdeg}(\varphi) \leq \langle i, q\rangle$, (ii) $\varphi \notin \mathrm{tp}_{i,q}(M_0, s_1, \overline{a}_0)$ (since by construction φ is falsified at $t_1 \subseteq s_1$) and (iii) $\varphi \in \mathrm{tp}_{i,q}(M_1, s_0, \overline{a}_1)$ (since by construction φ holds at every state $t_0 \subseteq s_0$). Thus we have $M_0, t_0, \overline{a}_0 \not\sqsubseteq_{i-1,q} M_1, t_1, \overline{a}_1$, as we wanted.

Case 2: the first move is a $\forall$-move. Suppose S starts by choosing $b_1 \in D_1$. As this is a winning strategy for S, for every choice $b_0 \in D_0$ of D we have

$$M_0, s_0, \overline{a}_0 b_0 \not\precsim_{i,q-1} M_1, s_1, \overline{a}_1 b_1$$

By induction hypothesis, the above translates to

$$\exists \psi_{b_0} \in \mathrm{tp}(b_0) \setminus \mathrm{tp}(b_1)$$

where $\mathrm{tp}(b_0) := \mathrm{tp}_{i,q-1}(M_0, s_0, \overline{a}_0 b_0)$ and $\mathrm{tp}(b_1) := \mathrm{tp}_{i,q-1}(M_1, s_1, \overline{a}_1 b_1)$.

Now the formula

$$\varphi := \forall x \bigvee\!\!\!\!\bigvee_{b_0 \in D_0} \psi_{b_0}$$

has IQ-degree at most $\langle i, q\rangle$, and by construction we have $\varphi \in \mathrm{tp}_{i,q}(M_0, s_0, \overline{a}_0)$ and $\varphi \notin \mathrm{tp}_{i,q}(M_1, s_1, \overline{a}_1)$. Thus, we have $M_0, t_0, \overline{a}_0 \not\sqsubseteq_{i-1,q} M_1, t_1, \overline{a}_1$.

Case 3: the first move is a $\overline{\exists}$-move. Reasoning as in the previous case, we find that there exists a $b_0 \in D_0$—the element chosen by S—s.t. for every $b_1 \in D_1$

$$\exists \theta_{b_1} \in \mathrm{tp}(t_0) \setminus \mathrm{tp}(t_1)$$

In particular, it follows that the formula

$$\varphi := \overline{\exists} x \bigwedge_{b_1 \in D^{M_1}} \psi_{b_1}$$

is a formula of complexity at most $\langle i, q\rangle$ such that $\varphi \in \mathrm{tp}_{i,q}(M_0, s_0, \overline{a}_0)$ and $\varphi \notin \mathrm{tp}_{i,q}(M_1, s_1, \overline{d_0})$. Again, it follows that $M_0, t_0, \overline{a}_0 \not\sqsubseteq_{i-1,q} M_1, t_1, \overline{a}_1$. □

As a corollary, we also get a game-theoretic characterization of the distinguishing power of formulas in the $\langle i, q\rangle$-fragment of InqBQ.

Corollary 1. *For a finite signature Σ, we have:*

$$M_0, s_0, \overline{a}_0 \approx_{i,q} M_1, s_1, \overline{a}_1 \quad \iff \quad M_0, s_0, \overline{a}_0 \equiv_{i,q} M_1, s_1, \overline{a}_1$$

3.4 Extending the result to function symbols

The results we just obtained assume that the signature Σ is relational. However, it is not hard to extend them to the case in which Σ contains function symbols (including nullary function symbols, i.e., constant symbols). In InqBQ, function symbols are interpreted rigidly: if $f \in \Sigma$ is an n-ary function symbol, then the interpretation function I of a model M must assign to all worlds w in the model the same function $I_w(f) : D^n \to D$.[6]

As in the case of classical logic [15], the presence of function requires some care in formulating the EF game. The fundamental reason is that allowing atomic formulas to contain arbitrary occurrences of function symbols allows us to generate with a finite number of choices in the game an infinite sub-structure of the model—which spoils the crucial locality feature of the game. Technically, a simple way to achieve this is to follow [15]§3.3 and work with formulas which are *unnested*.

Definition 4 (Unnested formula). *An* unnested atomic formula *is a formula of one of the following forms:*

$$x = y \qquad c = y \qquad f(\overline{x}) = \overline{y} \qquad R(\overline{x})$$

An unnested formula *is a formula that contains only unnested atoms.*

We can now make the following amendments to the definition above: (i) the winning conditions for the game are determined by looking at whether Equation (1) is satisfied for all *unnested* atomic formulas, and (ii) the $\langle i, q\rangle$-types are redefined as sets of *unnested* formulas of degree at most $\langle i, q\rangle$. Other than that, the statement of the result and the proof are the same as above.

Clearly, using identity we can turn an arbitrary formula into an equivalent unnested one (e.g., replacing $P(f(x))$ with $\forall y(y = f(x) \to Py)$) so the restriction to un-nested formula is not a limitation to the generality of the game-theoretic characterization; rather, it can be seen as an indirect way of assigning formulas containing function symbols with the appropriate $\langle i, q\rangle$-degree—making explicit a quantification which is implicit in the presence of a function symbol.

4 An application: expressive limitations of InqBQ

In this section we exploit our game-theoretic characterization of the expressive power of InqBQ to show that certain questions—i.e., certain global properties of information states—are not definable in InqBQ with respect to certain classes of structures.

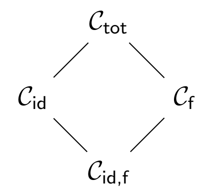
Let us start by making the relevant notions more precise. If $\mathcal{C}$ is a class of models for InqBQ, by a *property* over $\mathcal{C}$ we mean a sub-class $\mathcal{P} \subseteq \mathcal{C}$. We also write

[6] In the general case, non-rigid function symbols are also allowed; however, such symbols can be dispensed with as usual in favor of relation symbols constrained by suitable axioms. See §4.3.5 of [2] for the details.

$\mathcal{P}(M)$ instead of $M \in \mathcal{P}$.[7] Given a property $\mathcal{P}$ over $\mathcal{C}$, we say that a formula φ of InqBQ *defines* $\mathcal{P}$ over $\mathcal{C}$ if $\mathcal{P} = \{M \in \mathcal{C} \mid M \models \varphi\}$.

We will focus on the following classes:

- $\mathcal{C}_{\mathsf{tot}}$ the class of all inquisitive models;
- $\mathcal{C}_{\mathsf{f}}$ the class of finite models;
- $\mathcal{C}_{\mathsf{id}}$ the class of id-models;[8]
- $\mathcal{C}_{\mathsf{id,f}} = \mathcal{C}_{\mathsf{id}} \cap \mathcal{C}_{\mathsf{f}}$.



The figure on the right shows these classes ordered by containment. Notice that if a property $\mathcal{P}$ is definable by a formula φ relative to a class $\mathcal{C}$, and $\mathcal{C}' \subseteq \mathcal{C}$, then over $\mathcal{C}'$ the formula φ defines the restriction of $\mathcal{P}$ to $\mathcal{C}'$. Thus, a property which is definable over a class of models is also definable over a sub-class. Contrapositively, if a property $\mathcal{P}$ is not definable over $\mathcal{C}$ and $\mathcal{C}' \supseteq \mathcal{C}$, then no property $\mathcal{P}'$ whose restriction to $\mathcal{C}$ coincides with $\mathcal{P}$ can be definable over $\mathcal{C}'$. Thus, undefinability results are stronger when proved relative to smaller classes.

The first result that we show is that the question "how many individuals have property P"—for P a unary predicate symbol—is not definable in InqBQ, even in restriction to the class $\mathcal{C}_{\mathsf{id,f}}$ of finite id-models. To make this precise, recall that we denote by P_w the extension of property P at a world w. We denote by $\#P_w$ the cardinality of this set. Then the question "how many individuals have property P" is supported at a state s if the information available in s determines the cardinality of the set P_w, i.e., if $\forall w, w' \in s$ we have $\#P_w = \#P_{w'}$. The following result says that no sentence of InqBQ has these support conditions, even in restriction to finite id-models.

Theorem 2. *The following property is not definable over $\mathcal{C}_{\mathsf{id,f}}$:*

$$\mathcal{P}_1(M) \overset{def}{\iff} \forall w, w' \in W : \#P_w = \#P_{w'}$$

[7] Conceptually, it would be more natural to view InqBQ-formulae as defining properties $\mathcal{Q}$ of pointed models, i.e., pairs $\langle M, s\rangle$, where M is a model and s an information state in M. However, in the present context we are only interested in properties $\mathcal{Q}$ which are *local*, i.e., such that $\mathcal{Q}(M, s)$ holds iff $\mathcal{Q}(M \restriction s, s)$ holds, where $M \restriction s$ is the natural restriction of M to the set of worlds s. The reason for this restriction is that all properties of pointed models defined by InqBQ-formulas are local in this sense. Given the locality constraint, properties of pointed models and properties of models are in a 1-1 correspondence: given a property $\mathcal{P}$ of models, we can define a corresponding property $\mathcal{Q}_{\mathcal{P}}$ of pointed models by letting $\mathcal{Q}_{\mathcal{P}}(M, s) \iff \mathcal{P}(M \restriction s)$. Conversely, given a property $\mathcal{Q}$ of pointed models, we can define a corresponding property $\mathcal{P}_{\mathcal{Q}}$ of models by letting $\mathcal{P}_{\mathcal{Q}}(M) \iff \mathcal{Q}(M, W)$. We can thus work with properties of models without loss of generality, which is somewhat more convenient.

[8] Recall that an id-model is a model where $=$ is interpreted at each world as the identity relation on D: for all $w \in W$, $\sim_w = \{\langle d, d\rangle \mid d \in D\}$.

To make the argument clearer, we will consider only models in the signature $\{P^{(1)}\}$. The argument can be easily generalized to arbitrary signatures containing P.[9]

To show this result, a special class of models will be especially useful.

Definition 5 (switch models). *For $h, k \in \omega \cup \{\omega\}$, define the following model*[10]*:*

$$M_{h,k} = \langle \{w_0, w_1\}, [1, h+k], I\rangle$$

where $P_{w_0} = [1, h]$ and $P_{w_1} = [h+1, h+k]$.

We will start by showing an interesting property of these models. For two ordinals h, k and a natural number q, write $h =_q k$ in case $h = k$ or both h and k are $\geq q$.

Lemma 1. *Consider two switch models $M_{h,k}$ and $M_{h',k'}$ and $i, q \in \mathbb{N}$. If $h =_q h'$ and $k =_q k'$, then $M_{h,k} \equiv_{i,q} M_{h',k'}$.*

Proof. By Theorem 1 it suffices to give winning strategies for D in the two games with indexes $\langle i, q\rangle$ played between models $M_{h,k}$ and $M_{h',k'}$. Since the situation is symmetrical in the two cases, one direction suffices. The trick consists in preserving the following invariants through the game.

1. The info states in the current position are the same.
2. Given $\overline{a}$ and $\overline{b}$ the l-sequences of elements picked, then $a_i \leq h \iff b_i \leq h'$ and $a_i = a_j \iff b_i = b_j$ for every $i, j \leq l$.

Maintaining these invariants until the end of the game ensures the victory for D. The invariants can be maintained by the following strategy:

– If S plays an implication move and picks the state s in one of the models, then D picks s in the other model.
– If S plays a quantifier move and picks $a_l = a_i$ an element already chosen, then D picks $b_l = b_i$.
– If S plays a quantifier move and picks a_l a fresh element, then D picks b_l a fresh element such that $a_l \leq h \iff b_l \leq h'$. This is possible since $h =_q h'$ and $k =_q k'$ by hypothesis. □

We are now ready to prove Theorem 2.

Proof (Theorem 2). Suppose for a contradiction that there exists a φ which defines $\mathcal{P}_1$. Let $\langle i, q\rangle$ be its combined degree. Then by Lemma 1 the models $M_{q,q}$ and $M_{q,q+1}$ are $\langle i, q\rangle$-equivalent. In particular, $M_{q,q} \models \varphi \iff M_{q,q+1} \models \varphi$. But this contradicts the assumption that φ defines the property $\mathcal{P}_1$, since $\mathcal{P}_1$ holds for $M_{q,q}$ but not for $M_{q,q+1}$. □

[9] One way to generalize the result to an arbitrary signature Σ: extend the models in the proofs that follow by interpreting the symbols in $\Sigma \setminus \{P^{(1)}\}$ as the empty relation. The proofs then follow through.

[10] Where $+$ denotes ordinal sum and $[\alpha, \beta]$ denotes the set of ordinals $\{\gamma \mid \alpha \leq \gamma \leq \beta\}$.

With an analogous proofs we can show that the question "whether the number of elements that satisfy P is even" is not definable in InqBQ relative to the class of finite id-models. Generalizing slightly, for any natural k, we cannot express the question "what is the number of elements that satisfy P, modulo k". More formally, we have the following result.

Theorem 3. *The following property is not definable over $\mathcal{C}_{\mathsf{id,f}}$:*

$$\mathcal{P}_2(M) \overset{def}{\iff} \forall w, w' \in W : \#P_w \textit{ and } \#P_{w'} \textit{ are congruent modulo } k$$

Moving away from the restriction to finite domains, we can consider the question *whether the extension of P is finite* . This question is resolved in a state s in case from the information available in s it follows that the extension of P is finite (i.e., $\#P_w$ is finite for all $w \in s$) or it follows that the domain is infinite (i.e., $\#P_w$ is infinite for all $w \in s$). In other words, the question is resolved if all worlds in s agree on whether $\#P_w$ is finite. We can use switch models to show that this question is not definable in InqBQ, even in restriction to the class $\mathcal{C}_{\mathsf{id}}$.

Theorem 4. *The following property is not definable over $\mathcal{C}_{\mathsf{id}}$:*

$$\mathcal{P}_3(M) \overset{def}{\iff} \forall w, w' \in W : (\#P_w \textit{ is finite} \iff \#P_{w'} \textit{ is finite})$$

Proof. Suppose for a contradiction that some formula φ defines $\mathcal{P}_3$. Let $\langle i, q\rangle$ be its combined degree. By Lemma 1 the models $M_{q,q}$ and $M_{q,\omega}$ are $\langle i, q\rangle$-equivalent, and so in particular $M_{q,q} \models \varphi \iff M_{q,q+1} \models \varphi$. But this contradicts the assumption that φ defines $\mathcal{P}_3$, since $\mathcal{P}_3$ holds of $M_{q,q}$ but not of $M_{q,\omega}$. $\square$

Notice that, as we remarked above, these undefinability results extend to superclasses of the class for which they were stated: for instance, Theorem 2 implies that InqBQ cannot define any property of states s in arbitrary inquisitive model whose restriction to the class coincides with the property "the cardinality of P_w is constant in s".

Next, let us look at models where the interpretation of identity is not fixed, but varies across worlds. As we discussed in Section 2, the actual individuals existing at a world w can be equated with the equivalence classes modulo $\sim_w$, i.e., with the elements of the quotient $D/{\sim_w}$. Let us denote this quotient by D_w and refer to it as the *essential domain* at w. The number of actual individuals existing at w is the cardinality of this set, $\#D_w$. Now consider the question "how many individuals there are". This question is resolved at a state s if in s there is no uncertainty about the cardinality of the essential domain, i.e., if all the worlds $w \in s$ agree on the number $\#D_w$. The following theorem says that this question is not definable in InqBQ, even in restriction to the class $\mathcal{C}_{\mathsf{f}}$ of finite models.

Theorem 5. *The following property is not definable over $\mathcal{C}_{\mathsf{f}}$:*

$$\mathcal{P}_4(M) \models \varphi \overset{def}{\iff} \forall w, w' \in W : \#D_w = \#D_{w'}$$

As before we will only consider models in the empty signature—thus we have to deal only with atoms of the form $(t = t')$—but the theorem can be easily generalized to arbitrary signatures.

To prove this result we will introduce another useful class of models.

Definition 6 (Grid models). *For $h, k \in \omega \cup \{\omega\}$, define the model*

$$G_{h,k} = \langle \{w_0, w_1\}, [1,h] \times [1,k], I \rangle$$

where, denoting by $\sim_w$ the equivalence relation $I_w(=)$, we have:

$$\langle a, b\rangle \sim_{w_0} \langle a', b'\rangle \iff a = a'$$
$$\langle a, b\rangle \sim_{w_1} \langle a', b'\rangle \iff b = b'$$

Given two sequences of elements $\overline{a} = \langle a_1; \dots; a_l\rangle$ and $\overline{b} = \langle b_1; \dots; b_l\rangle$, define $\mathbf{zip}(\overline{a}, \overline{b}) = \langle\langle a_1, b_1\rangle; \dots; \langle a_l, b_l\rangle\rangle$.

Notice that, in a grid model $G_{h,k}$, the essential domain D_{w_0} has cardinality h, while the essential domain D_{w_1} has cardinality k. Thus, the cardinality of the essential domain is constant in $G_{h,k}$ only if $k = h$.

Definition 7. *Given $\overline{a}$ a sequence of natural numbers, we define*

$$E_{\overline{a}} = \{\langle i, j\rangle \mid a_i = a_j\}$$
$$E_{\overline{a}}|_l = E_{\overline{a}} \cap ([1,l] \times [1,l])$$

Proposition 3. *Consider a grid model $G_{h,k}$, a state s and a sequence of elements $\mathbf{zip}(\overline{a}, \overline{b})$. Then $\mathrm{tp}_{0,0}(G_{h,k}; s; \mathbf{zip}(\overline{a}, \overline{b}))$ is univocally determined by the sets $E_{\overline{a}} = \{\langle i, j\rangle \,|a_i = a_j\}$ and $E_{\overline{b}} = \{\langle i, j\rangle \,|b_i = b_j\}$.*

Proposition 4. *Fix $l, h, k \in \mathbb{N}$ such that $l < \min(h, k)$. Consider a grid model $G_{h,k}$ and two sequences of elements $\mathbf{zip}(\overline{a}, \overline{b})$ and $\mathbf{zip}(\overline{c}, \overline{d})$ of length l and $l + 1$ respectively. Moreover suppose that*

$$E_{\overline{a}} = E_{\overline{c}}|_l \qquad\qquad E_{\overline{b}} = E_{\overline{d}}|_l$$

Then there exists an element $\langle a_{l+1}, b_{l+1}\rangle$ such that

$$E_{\overline{a}a_{l+1}} = E_{\overline{c}} \qquad\qquad E_{\overline{b}b_{l+1}} = E_{\overline{d}}$$

Proof. First we choose a_{l+1} conditionally on $\overline{c}$: if $c_{l+1} = c_i$ for $i < l$, then pick $a_{l+1} = a_i$; otherwise pick an element in $[1, h] \setminus \{a_1, \dots, a_l\}$—notice this is always possible as $l < h$. b_{l+1} is chosen in a similar way conditionally on $\overline{d}$. The identities $E_{\overline{a}a_{l+1}} = E_{\overline{c}}$ and $E_{\overline{b}b_{l+1}} = E_{\overline{d}}$ hold by construction. □

The following Lemma is an analogue of Lemma 1 for switch models.

Lemma 2. *Consider $G_{h,k}$ and $G_{h',k'}$ two grid models and $i, q \in \mathbb{N}$. If $h =_q h'$ and $k =_q k'$, then $G_{h,k} \equiv_{i,q} G_{h',k'}$.*

Proof. We will define a winning strategy for D applicable to both the games $\mathrm{EF}_{i,q}(G_{h,k}, G_{h',k'})$ and $\mathrm{EF}_{i,q}(G_{h',k'}, G_{h,k})$. The trick of the strategy consists in preserving the following invariants through the game.

1. The info states in the current position are the same.
2. Given $\mathbf{zip}(\overline{a}, \overline{b})$ and $\mathbf{zip}(\overline{a'}, \overline{b'})$ the sequences of elements in the current position, it holds $E_{\overline{a}} = E_{\overline{a'}}$ and $E_{\overline{b}} = E_{\overline{b'}}$.

By Proposition 3, preserving the invariants ensures the victory of player D. The winning strategy for D is the following:

– If S plays a $\rightarrow$-move and chooses s: D chooses s in the other model. This ensures the invariant is preserved.
– If S plays a $\forall$ or $\overline{\exists}$ move and chooses $\langle a_k, b_k\rangle$ from one model: Suppose the current sequences of elements are $\mathbf{zip}(\overline{a}, \overline{b})$ and $\mathbf{zip}(\overline{a}', \overline{b}')$. Condition 2 ensures that $E_{\overline{a}} = E_{\overline{a}'}$ and $E_{\overline{b}} = E_{\overline{b}'}$. By Proposition 4, D can choose an element $\langle a'_k, b'_k\rangle$ such that $E_{\overline{a}a_k} = E_{\overline{a}'a'_k}$ and $E_{\overline{b}b_k} = E_{\overline{b}'b'_k}$. This ensures the invariant is preserved. □

We are now ready to prove Theorem 5.

Proof (Theorem 5). Suppose for a contradiction that some formula φ of InqBQ defines $\mathcal{P}_4$. Let $\langle i, q\rangle$ be the combined degree of φ. By Lemma 2 the models $G_{q,q}$ and $G_{q,q+1}$ are $\langle i, q\rangle$-equivalent, so $G_{q,q} \models \varphi \iff G_{q,q+1} \models \varphi$. But this contradicts the assumption that φ defines the property $\mathcal{P}_4$, since $\mathcal{P}_4$ holds of $G_{q,q}$ but not of $G_{q,q+1}$. □

In a similar way we can prove that, relative to finite models, InqBQ cannot express the question "how many individuals there are, modulo k" (and, as a special case, "whether the number of individuals is even"). This is formalized by the following theorem.

Theorem 6. *The following property is not definable over $\mathcal{C}_{\mathsf{f}}$:*

$$\mathcal{P}_5(M) \models \varphi \overset{def}{\iff} \forall w, w' \in W : \#D_w \textit{ and } \#D_{w'} \textit{ are congruent modulo } k$$

We can also use grid models to show that, this time relative to the class $\mathcal{C}_{\mathsf{tot}}$ of all models, InqBQ cannot express the question "whether the domain is finite", which is supported in a state s in case all worlds in s agree on whether the domain is finite. The proof is analogous to the one of Theorem 4, using grid models instead of switch models.

Theorem 7. *The following property is not definable over $\mathcal{C}_{\mathsf{tot}}$:*

$$\mathcal{P}_6(M) \overset{def}{\iff} \forall w, w' \in W : (\#D_w \textit{ is finite} \iff \#D_{w'} \textit{ is finite})$$

Finally, notice that, in the general case where the interpretation of identity varies across worlds, the number of actual individuals that satisfy P at a world is not given by $\#P_w$, but rather by the number of equivalence classes $\#(P_w/\sim_w)$. So, in the general setting of $\mathcal{C}_{\mathsf{tot}}$ (or $\mathcal{C}_{\mathsf{f}}$), the question "how many individuals have property P" corresponds to the following property:

$$\mathcal{P}_7(M) \overset{def}{\iff} \forall w, w' \in W : \#(P_w/\sim_w) = \#(P_{w'}/\sim_{w'})$$

However, notice that the restriction of this property to the class $\mathcal{C}_{\mathsf{id,f}}$ of finite id-models is just the property $\mathcal{P}_1$. Since $\mathcal{P}_1$ is not definable over $\mathcal{C}_{\mathsf{id,f}}$, it follows that $\mathcal{P}_7$ is not definable over $\mathcal{C}_{\mathsf{tot}}$ (or $\mathcal{C}_{\mathsf{f}}$). Analogous conclusions can be drawn for the generalized versions of properties $\mathcal{P}_2$ and $\mathcal{P}_3$ which take into account the role of variable identity: in restriction to id-models, the generalized version boils down to the simple version considered above; since the simple version is not definable, the generalized version is not definable either.

5 Conclusions and further work

EF games often provide an insightful perspective on a logic, and a useful characterization of its expressive power. In this paper we have described an EF game for inquisitive first-order logic, InqBQ. This game presents two novelties with respect to its classical counterpart. First, the roles of the two models on which the game is played are not symmetric: certain moves have to be performed mandatorily in one of the models. This feature reflects the fact that InqBQ lacks a classical negation, and that the theory of a model—unlike in the classical case—can be properly included in that of another. Secondly, the objects that are picked in the course of the game are not just individuals $d \in D$, but also information states, i.e., subsets $s \subseteq W$ of the universe of possible worlds. This feature reflects the fact that InqBQ contains not only the quantifiers $\forall, \overline{\exists}$ over individuals, but also the implication $\to$, which allows for a restricted kind of quantification over information states. We proved that certain fragments of the language of InqBQ are preserved from a model M_0 to a model M_1 if and only if a winning strategy exists for Duplicator in a corresponding finite game played on these models. We used this result to establish a number of undefinability results. In particular, we have seen that the question *how many individuals satisfy* P (or any variant of this question concerning cardinality modulo $k > 1$) is not expressible, even in restriction to finite models in which '=' is interpreted rigidly as the actual identity on D. Moreover, the question *whether the extension of* P *is finite* is not expressible, even given this restriction on the interpretation of identity. In the general context where the interpretation of identity is variable across worlds, analogous results hold concerning the total number of individuals in the domain.

The work presented in this paper can be taken further in several directions. First, in the context of classical logic, several variants of the EF game have been studied. For example, [23] presents a *dynamic* EF game, corresponding to a more fine-grained classification of classical structures. In the inquisitive case, an analogous refinement could lead to interesting insights into the structure of inquisitive models.

Second, EF games can be used to study which properties invariant under automorphism are expressible in InqBQ. This includes, in particular, properties dependent only on the cardinality of the domain: $\mathcal{P}_4$, $\mathcal{P}_5$ and $\mathcal{P}_6$ are *not* expressible in InqBQ, while, e.g., "there are n individuals" for $n \in \mathbb{N}$ is expressible. In future work, we aim to use the game-theoretic perspective developed in this paper to study in more generality which automorphism-invariant properties—in particular, which cardinality properties—are expressible in InqBQ.

Finally, EF games can be used also to compare different extensions of a fixed logic, as shown in [17]. In this regard, the results presented in Section 4 already yield some interesting corollaries. For example, adding to InqBQ a generalized quantifier corresponding to any of the properties $\mathcal{P}_1, \ldots, \mathcal{P}_7$ discussed in Section 4 yields a logic which is strictly more expressive than InqBQ. More generally, the techniques introduced in this paper are likely to provide a useful tool for a systematic study of generalized quantifiers in inquisitive logic.

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