Exercise 1 (Partially decidable predicates)**.**
Are the following predicates partially decidable? In each case, give a proof.

1. $M_1(x) = \neg M_{\text{tot}}(x) =$ "ϕ_x is non-total"
2. $M_2(x) =$ "$\text{Ran}(\phi_x)$ includes an even number"
3. $M_3(x) =$ "$\text{Ran}(\phi_x)$ is finite"

Solutions.

1. M_1 is not partially decidable. To see this, we reduce the predicate $M_{\overline{K}} =$ "$\phi_x(x)\!\uparrow$" to M_1.

   For the reduction, we use same function that we have often used:

   $$f(x,y) = \begin{cases} 0 & \text{if } \phi_x(x)\!\downarrow \\ \text{undefined} & \text{if } \phi_x(x)\!\uparrow \end{cases}$$

   As usual, we use the s-m-n theorem to obtain a total computable $k : \mathbb{N} \to \mathbb{N}$ such that $\phi_{k(x)}(y) = f(x,y)$. It is easy to verify that for any $x \in \mathbb{M}$, $M_{\overline{K}}(x) \iff M_1(k(x))$. So, we have indeed reduced $M_{\overline{K}}$ to M_1; since the former is not p.d., neither is the latter.

2. M_2 is partially decidable. To see this, we note that we have:

   $$M_2(x) \iff \exists y (y \in \text{Ran}(\phi_x) \text{ and } y \text{ is even})$$

   We know that the predicate "y is even" is decidable, and thus partially decidable; and we have seen in class that the predicate "$y \in \text{Ran}(\phi_x)$" is partially decidable. In one of the previous exercises, we have also shown that, if M and M' are p.d., then so is the conjunction M and M'. So, the conjunctive predicate "$y \in \text{Ran}(\phi_x)$ and y is even" is partially decidable.

   Finally, since M_2 is obtained by existentially quantifying over a partially decidable predicate, it is itself partially decidable.

3. M_3 is not partially decidable. To see this, we reduce again $M_{\overline{K}} =$ "$\phi_x(x)\!\uparrow$" to M_3.

   First, define a predicate $M^*(x,y) =$ "P_x halts on x in at most y steps". This predicate is decidable, by CT. Then we define the following function:

   $$f(x,y) = \begin{cases} y & \text{if } M^*(x,y) \text{ holds} \\ \text{undefined} & \text{if } M^*(x,y) \text{ doesn't hold} \end{cases}$$

   Since M^* is decidable, this function is computable. As usual, we use the s-m-n theorem to extract a function $k : \mathbb{N} \to \mathbb{N}$ such that $\phi_{k(x)}(y) = f(x,y)$. Let us check that this function performs the desired reduction, i.e., that for any $x \in \mathbb{M}$: $M_{\overline{K}}(x) \iff M_3(k(x))$.

- Suppose $M_{\overline{K}}(x)$ holds. This means that $\phi_x(x)\uparrow$, so there is no y such that $M^*(x,y)$ holds. Thus, for all y we have $\phi_{k(x)}(y)\uparrow$. This means that $\mathrm{Ran}(\phi_{k(x)}) = \emptyset$, and $M_3(k(x))$ holds.
- Suppose $M_{\overline{K}}(x)$ doesn't hold. This means that $\phi_x(x)\downarrow$. Let y_0 be the number of steps in which P_x halts on x. Then for all $y \geq y_0$, $M^*(x,y)$ holds, and therefore $\phi_{k(x)}(y) = y$. Hence, for all $y \geq y_0$ we have $y \in \mathrm{Ran}(\phi_{k(x)})$. Thus, $\mathrm{Ran}(\phi_{k(x)})$ is infinite, and $M_3(k(x))$ doesn't hold.

This proves that $M_{\overline{K}}$ reduces to M_3 via the function k. Since $M_{\overline{K}}$ is not partially decidable, neither is M_3.