Exercises on reduction and undecidability

Exercise 1.
Show that the following predicates are undecidable:

1. $M_1(x, y) =$ "$\mathrm{Dom}(\phi_x) = \mathrm{Dom}(\phi_y)$"
2. $M_2(x) =$ "$\phi_x(x) = 0$"
3. $M_3(x, y) =$ "$\phi_x(y) = 0$"
4. $M_4(x, y) =$ "$x \in \mathrm{Ran}(\phi_y)$"
5. $M_5(x) =$ "ϕ_x is total and constant"
6. $M_6(x) =$ "$\mathrm{Dom}(\phi_x) = \emptyset$"
7. $M_7(x) =$ "$\mathrm{Ran}(\phi_x)$ is infinite"
8. $M_8(x) =$ "$\phi_x = g$" where g is a fixed computable function

Solutions.

1. Let e be an index for the constant zero function. This means that $\phi_e = \mathbb{O}$, and so $\mathrm{Dom}(\phi_e) = \mathbb{N}$. Let $M_{\mathrm{tot}}(x) =$ "$\phi_x$is total". We have:

   $$M_{\mathrm{tot}}(x) \iff \mathrm{Dom}(\phi_x) = \mathbb{N} \iff \mathrm{Dom}(\phi_x) = \mathrm{Dom}(\phi_e) \iff M_1(x, e)$$

   This shows that if we could decide M_1, we could also decide M_{tot}. But we know that the latter is not decidable. Hence, M_1 is not decidable either.

2. To show that M_2 is undecidable, we use the diagonal method. Consider the function:

   $$f(x) = \begin{cases} 1 & \text{if } \phi_x(x) = 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } M_2(x) \text{ holds} \\ 0 & \text{otherwise} \end{cases}$$

   Notice that f is the characteristic function of $M_2(x)$. Now, f cannot be equal to any computable function ϕ_x, since by construction $f(x) \neq \phi_x(x)$. So, f is undecidable. Since f is the characteristic function of M_2, this means that M_2 is undecidable.

3. Since $M_2(x) = M_3(x, x)$, if M_3 was decidable so would be M_2. But we have just seen that M_2 is undecidable.

4. This was exercise 7 of the last exam.

5. Let $\mathcal{B} = \{f \in \mathcal{C}^{(1)} \mid f \text{ is total and constant}\}$. Clearly, $M_5(x) =$ "$\phi_x \in \mathcal{B}$". Since $\mathcal{B}$ contains some but not all unary computable functions, Rice's theorem applies, ensuring that M_5 is undecidable.

6. **Strategy 1.** Let $f_\emptyset$ denote the function which is always undefined. Notice that $f_\emptyset$ is the only function with an empty domain, that is, $\mathrm{Dom}(\phi_x) = \emptyset \iff \phi_x = f_\emptyset$. So, if we let $\mathcal{B} = \{f_\emptyset\}$, then we have: $M_6(x) \iff \mathrm{Dom}(\phi_x) = \emptyset \iff \phi_x = f \iff \phi_x \in \mathcal{B}$. Since $\mathcal{B}$ is a non-empty and non-total class of computable functions, Rice's theorem applies. So, M_6 is undecidable.

   Strategy 2. We can get to the same result by using the reduction method. We do this by reducing the undecidable predicate $M_{\overline{K}}(x) =$ "$\phi_x(x)\uparrow$" to M_6. For this, we define a function $f(x, y)$ as follows:

$$f(x,y) = \begin{cases} 0 & \text{if } \phi_x(x)\downarrow \\ \text{undefined} & \text{if } \phi_x(x)\uparrow \end{cases}$$

   This function is computable by Church's thesis (a procedure to compute f should be given for this, but this procedure goes in the usual manner). By the s-m-n theorem, we have a total computable k such that $\phi_{k(x)}(y) = f(x, y)$.

   We claim that $M_{\overline{K}} \leq^k M_6$, that is, that $\forall x \in \mathbb{N}$, $M_{\overline{K}}(x) \iff M_6(k(x))$. If this claim is true, since $M_{\overline{K}}$ is undecidable, it follows that M_6 is undecidable as well. So, we are left with the task of showing the claim.

   - Suppose $M_{\overline{K}}(x)$ holds. This means that $\phi_x(x)\uparrow$. So, for all y, $\phi_{k(x)}(y)\uparrow$. Hence, $\mathrm{Dom}(\phi_{k(x)}) = \emptyset$, which means that $M_6(k(x))$ holds.
   - Suppose $M_{\overline{K}}(x)$ doesn't hold. This means that $\phi_x(x)\downarrow$. So, for all y, we have $\phi_{k(x)}(y) = 0$. Hence, $\mathrm{Dom}(\phi_{k(x)}) = \mathbb{N}$, which means that $M_6(k(x))$ does not hold.

7. As for the previous item, we have at least two strategies. First, we can observe that $M_7 = M_{\mathcal{B}}$ for $\mathcal{B} = \{f \in \mathcal{C}^{(1)} \mid \mathrm{Ran}(f) \text{ is infinite}\}$, and we can directly apply Rice's theorem.

   Second, we can reduce $M_K =$ "$\phi_x(x)\downarrow$" to M_7 using a strategy similar to that in the previous exercise. In particular, we could start with the function $f(x, y)$ defined as follows

$$f(x,y) = \begin{cases} y & \text{if } \phi_x(x)\downarrow \\ \text{undefined} & \text{if } \phi_x(x)\uparrow \end{cases}$$

   and then proceed in a way similar to the one described in the previous exercise. Both strategies were discussed in detail in class.

8. M_8 is the specification problem, i.e., the problem of deciding, for a fixed function, whether a given program computes the function or not. The undecidability of this problem was proved in class as a prominent corollary of Rice's theorem.

   [A direct proof by reduction is also possible. This can be modeled roughly after the proof of Rice's theorem.]