2

INTUITIONISTIC LOGIC

From the set-theoretic point of view intuitionistic propositional logic is a subset of the classical one: it can be defined by the calculus which is obtained from *Cl* by discarding the law of the excluded middle (A10). It is Brouwer's (1907, 1908) criticism of this law that intuitionistic logic stems from. However, the philosophical and mathematical justifications of these two logics are fundamentally different.

2.1 Motivation

The law of the excluded middle allows proof of disjunctions $\varphi \vee \psi$ such that neither φ nor ψ is provable. It is equivalent in **Cl** to the formula $\neg\neg p \to p$ justifying proofs by *reductio ad absurdum*, which make it possible to prove the existence of an object (having some given properties) without showing a way of constructing it. Proofs of that sort are known as *non-constructive*. The aim of intuitionistic logic is to single out and describe the laws of "constructive" reasoning.

The main principle of intuitionism asserts that the truth of a mathematical statement can be established only by producing a constructive proof of the statement. So the intended meaning of the intuitionistic logical connectives is defined in terms of *proofs* and *constructions*. The notions "proof" and "construction" themselves are regarded as primary, and it is assumed that we understand what a proof of an atomic proposition is.

- A proof of a proposition $\varphi \wedge \psi$ consists of a proof of φ and a proof of ψ.
- A proof of $\varphi \vee \psi$ is given by presenting either a proof of φ or a proof of ψ.
- A proof of $\varphi \to \psi$ is a construction which, given a proof of φ, returns a proof of ψ.
- $\perp$ has no proof and a proof of $\neg\varphi$ is a construction which, given a proof of φ, would return a proof of $\perp$.

This interpretation, given by Brouwer, Kolmogorov[3] (1932) and Heyting (1956), can hardly be reckoned as a precise semantic definition and used for constructing intuitionistic logic, as it was done for **Cl**. Nevertheless, it is not difficult to see that the first nine axioms of classical calculus *Cl* are entirely acceptable from the intuitionistic point of view, while the law of the excluded middle must be

[3]Kolmogorov treated formulas as schemes of solving (or posing) problems; for example, $\varphi \to \psi$ means the problem: given any solution to the problem φ, find a solution to the problem ψ.

rejected (indeed, we cannot present now a proof of Goldbach's conjecture or that $\boldsymbol{P} = \boldsymbol{NP}$, etc., nor are we able to show that these statements do not hold).

Intuitionistic logic was first constructed in the form of calculus by Heyting (1930). This calculus (an equivalent one, to be more exact) is obtained from Cl by discarding axiom (A10).

As to the interpretation above, it can be made more precise in various ways. Two of them—Kleene's realizability interpretation and Medvedev's finite problem interpretation—will be briefly discussed in Section 2.9. Another way, connected with the explicit introduction of a new provability operator, will be considered in Section 3.9 of Chapter 3 dealing with modal logic.

More suitable for the practical use strict and philosophically significant definitions of semantics for intuitionistic logic were given by Beth (1956) and Kripke (1965a) (see also Grzegorczyk, 1964). Their semantics does not exploit the notions of proof and construction; instead, it explicitly expresses an epistemic feature of intuitionistic logic. We will give now some informal motivation of the Kripke semantics; the corresponding formal definitions will be introduced in the next section.

By accepting the fundamental semantic assumption of classical logic—each proposition is either true or false—we completely abstract from the fact that actually it may be *a priori* unknown whether this or that proposition is true or false. We do not know now, for instance, if Goldbach's conjecture is true, if the equality $\boldsymbol{P} = \boldsymbol{NP}$ holds, whether there are rational beings in the Archer constellation, and so forth. But it is quite possible that we can know about this in the future, acquiring new information on mathematics and the world around us.

It is this epistemic aspect of the notion of truth that intuitionistic logic, as opposed to the classical one, takes into account.

Let us imagine that our knowledge is developing discretely, nondeterministically passing from one state to another. When at some state of knowledge (or information) x, we can say which facts are known at x and which are not established yet. Besides, we know what states of information y are possible in the future. Of course, this does not mean that we shall necessarily reach all these possible states (for instance, we can imagine now not only a course of events under which Goldbach's conjecture will be proved, but also such a situation when it will remain unproved or will be refuted). It is reasonable also to assume that while passing to a new state y all the facts known at x will be preserved, and some new facts will possibly be established.

It is natural to regard an atomic proposition, established at a state x, to be true at x; it will remain true at all further possible states. A proposition which is not true at x cannot be in general regarded as false, for it may become true at one of the subsequent states.

The truth of compound propositions can be defined now as follows.

- $\varphi \wedge \psi$ is true at a state x if both φ and ψ are true at x.
- $\varphi \vee \psi$ is true at x if either φ or ψ is true at x.

- $\varphi \to \psi$ is true at a state x if, for every subsequent possible state y, in particular x itself, φ is true at y only if ψ is true at y.
- $\bot$ is true nowhere.

It follows from this definition that the negation $\neg\varphi = \varphi \to \bot$ is true at x if φ is true at no subsequent possible state. A proposition φ may be regarded to be false at x if $\neg\varphi$ is true at x.

All axioms (A1)–(A9) (under every substitution of concrete propositions instead of variables) turn out to be true at all conceivable states, which cannot be said about (A10), i.e., $p_0 \vee (p_0 \to \bot)$. Indeed, if a proposition φ is not true at a state x, but becomes true at a subsequent state y, then $\neg\varphi$ is not true at x and so neither is $\varphi \vee \neg\varphi$.

2.2 Kripke frames and models

As in Section 1.1, let us fix the propositional language $\mathcal{L}$ with the connectives $\wedge$, $\vee$, $\to$ and the constant $\bot$. Starting from the informal interpretation above, we give now a precise definition of an intuitionistic model for $\mathcal{L}$.

An *intuitionistic Kripke frame* is a pair $\mathfrak{F} = \langle W, R\rangle$ consisting of a nonempty set W and a partial order R on W, i.e., $\mathfrak{F}$ is just a partially ordered set. We remind the reader that a binary relation R on W is called a *partial order* if the following three conditions[4] are satisfied for all $x, y, z \in W$:

$$\begin{array}{ll} xRx & \text{(reflexivity)}, \\ xRy \wedge yRz \to xRz & \text{(transitivity)}, \\ xRy \wedge yRx \to x = y & \text{(antisymmetry)}. \end{array}$$

The elements of W are called the *points* of the frame $\mathfrak{F}$ and xRy is read as "y is accessible from x" or "x sees y".

A *valuation* of $\mathcal{L}$ in an intuitionistic frame $\mathfrak{F} = \langle W, R\rangle$ is a map $\mathfrak{V}$ associating with each variable $p \in \mathbf{Var}\mathcal{L}$ some (possibly empty) subset $\mathfrak{V}(p) \subseteq W$ such that, for every $x \in \mathfrak{V}(p)$ and $y \in W$, xRy implies $y \in \mathfrak{V}(p)$. Subsets of W satisfying this condition are called *upward closed*. The set of all upward closed subsets of W will be denoted by $\mathrm{Up}W$. Thus, a valuation in $\mathfrak{F}$ is a map $\mathfrak{V}$ from $\mathbf{Var}\mathcal{L}$ into $\mathrm{Up}W$.

An *intuitionistic Kripke model* of the language $\mathcal{L}$ is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ where $\mathfrak{F}$ is an intuitionistic frame and $\mathfrak{V}$ a valuation in $\mathfrak{F}$.

In the terminology of the preceding section points in a frame $\mathfrak{F} = \langle W, R\rangle$ of a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ represent states of information; if we are now at a state x then in the sequel we may reach a state y such that xRy. An atomic proposition p is regarded to be true at x if $x \in \mathfrak{V}(p)$. Since $\mathfrak{V}(p)$ is upward closed, all atomic propositions that are true at x remain true at all subsequent possible states.

[4]Here and below, to represent various properties of frames we use the language of classical predicate logic with the predicates R and $=$.

Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ be an intuitionistic Kripke model and x a point in the frame $\mathfrak{F} = \langle W, R \rangle$. By induction on the construction of a formula φ we define a relation $(\mathfrak{M}, x) \models \varphi$, which is read as "$\varphi$ is true at x in $\mathfrak{M}$":

$$\begin{array}{lll}
(\mathfrak{M}, x) \models p & \text{iff} & x \in \mathfrak{V}(p); \\
(\mathfrak{M}, x) \models \psi \wedge \chi & \text{iff} & (\mathfrak{M}, x) \models \psi \text{ and } (\mathfrak{M}, x) \models \chi; \\
(\mathfrak{M}, x) \models \psi \vee \chi & \text{iff} & (\mathfrak{M}, x) \models \psi \text{ or } (\mathfrak{M}, x) \models \chi; \\
(\mathfrak{M}, x) \models \psi \rightarrow \chi & \text{iff} & \text{for all } y \in W \text{ such that } xRy, \\
& & (\mathfrak{M}, y) \models \psi \text{ implies } (\mathfrak{M}, y) \models \chi; \\
(\mathfrak{M}, x) \not\models \bot. & &
\end{array}$$

It follows from this definition that

$$(\mathfrak{M}, x) \models \neg\psi \quad \text{iff} \quad \text{for all } y \in W \text{ such that } xRy,\ (\mathfrak{M}, y) \not\models \psi.$$

If $\mathfrak{M}$ is understood we write $x \models \varphi$ instead of $(\mathfrak{M}, x) \models \varphi$. The truth-set of φ in $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$, i.e., the set $\{x : x \models \varphi\}$, will be denoted by $\mathfrak{V}(\varphi)$.

Notice that an intuitionistic model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ on the frame $\mathfrak{F}$ containing only a single point, say x, is in essence the same as the classical model

$$\mathfrak{N} = \{p \in \mathbf{Var}\mathcal{L} : x \in \mathfrak{V}(p)\},$$

because $(\mathfrak{M}, x) \models \varphi$ iff $\mathfrak{N} \models \varphi$, for every formula φ.

Proposition 2.1 *For every intuitionistic Kripke model on a frame $\mathfrak{F} = \langle W, R \rangle$, every formula φ and all points $x, y \in W$, if $x \models \varphi$ and xRy then $y \models \varphi$.*

Proof An easy induction on the construction of φ is left to the reader as an exercise. ❑

In other words, Proposition 2.1 states that the set of points where φ is true is upward closed. On the contrary, the set of points at which φ is not true may be called *downward closed*, since $x \not\models \varphi$ and yRx imply $y \not\models \varphi$.

We say a formula φ is *satisfied* in a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ if $x \models \varphi$ for some point x in $\mathfrak{F}$. φ is *true* in $\mathfrak{M}$ if $x \models \varphi$ for every x in $\mathfrak{F}$; in this case we write $\mathfrak{M} \models \varphi$. If φ is not true in $\mathfrak{M}$ then we say that φ is *refuted* in $\mathfrak{M}$ or $\mathfrak{M}$ is a *countermodel* for φ, and write $\mathfrak{M} \not\models \varphi$.

A formula φ is *satisfied* in a frame $\mathfrak{F}$ if φ is satisfied in some model based on $\mathfrak{F}$. φ is *true at a point* x in $\mathfrak{F}$ (notation: $(\mathfrak{F}, x) \models \varphi$) if φ is true at x in every model based on $\mathfrak{F}$. φ is called *valid* in a frame $\mathfrak{F}$, $\mathfrak{F} \models \varphi$ in symbols, if φ is true in all models based on $\mathfrak{F}$. Otherwise we say that φ is *refuted* in $\mathfrak{F}$ and write $\mathfrak{F} \not\models \varphi$.

If every formula in a set Γ is true at a point x in a model $\mathfrak{M}$, we write $(\mathfrak{M}, x) \models \Gamma$ or simply $x \models \Gamma$. $\mathfrak{M} \models \Gamma$ and $\mathfrak{F} \models \Gamma$ mean that all formulas in Γ are true in $\mathfrak{M}$ and are valid in $\mathfrak{F}$, respectively.

Frames $\mathfrak{F} = \langle W, R \rangle$ and $\mathfrak{G} = \langle V, S \rangle$ are said to be *isomorphic* if there is a 1–1 map f from W onto V such that xRy iff $f(x)Sf(y)$, for all $x, y \in W$. The map f is called then an *isomorphism* of $\mathfrak{F}$ onto $\mathfrak{G}$. Models $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ and $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$

are *isomorphic* if there is an isomorphism f of $\mathfrak{F}$ onto $\mathfrak{G}$ such that, for every $p \in \mathbf{Var}\mathcal{L}$, $\mathfrak{U}(p) = f(\mathfrak{V}(p))$, i.e., for every $x \in W$,

$$(\mathfrak{M}, x) \models p \text{ iff } (\mathfrak{N}, f(x)) \models p.$$

In this case we say that f is an *isomorphism* of $\mathfrak{M}$ onto $\mathfrak{N}$.

The following two propositions are direct consequences of the given definitions.

Proposition 2.2 *If f is an isomorphism of a model $\mathfrak{M}$ onto a model $\mathfrak{N}$ then, for every point x in $\mathfrak{M}$ and every formula φ,*

$$(\mathfrak{M}, x) \models \varphi \textit{ iff } (\mathfrak{N}, f(x)) \models \varphi.$$

This gives us the ground not to distinguish between isomorphic models as well as isomorphic frames.

Proposition 2.3 *Suppose $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ and $\mathfrak{N} = \langle \mathfrak{F}, \mathfrak{U} \rangle$ are models on a frame $\mathfrak{F}$ such that the valuations $\mathfrak{V}$ and $\mathfrak{U}$ coincide on the variables in some set* $\mathbf{Var} \subseteq \mathbf{Var}\mathcal{L}$*. Then for every point x in $\mathfrak{F}$ and every formula φ with* $\mathbf{Var}\varphi \subseteq \mathbf{Var}$*,*

$$(\mathfrak{M}, x) \models \varphi \textit{ iff } (\mathfrak{N}, x) \models \varphi.$$

Thus, if we want to construct a countermodel for a formula φ on a frame $\mathfrak{F}$, it suffices to define a valuation $\mathfrak{V}$, refuting φ, only on the variables in φ; the values of $\mathfrak{V}$ on other variables have no effect on the truth of φ at points in $\mathfrak{F}$.

We shall often represent intuitionistic frames in the form of diagrams by depicting points as circles $\circ$ and drawing an arrow from x to y if xRy. To avoid awkwardness, we will not draw those arrows that can be uniquely reconstructed by the properties of reflexivity and transitivity. For technical reasons it is sometimes impossible to connect x and y with an arrow; we then connect them with a (broken) line, and the fact that xRy is reflected by placing y higher than x. When representing models, we shall sometimes write some formulas near points: on the left side of a point x we write those formulas that are true at x and those that are not true are written on the right.

Example 2.4 Suppose $\mathfrak{F} = \langle W, R \rangle$ is the frame in which $W = \{a, b\}$, $R = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, b \rangle\}$ and let $\mathfrak{V}(p) = \{b\}$ and $\mathfrak{V}(q) = \{a, b\}$ for all $q \in \mathbf{Var}\mathcal{L}$ different from p. Then the formula $p \vee (p \to \bot)$ is true at b and not true at a in the model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$. This situation is represented graphically in Fig. 2.1. Thus, $p \vee (p \to \bot)$ is satisfied as well as refuted in $\mathfrak{F}$. The formula $((p \to \bot) \to \bot) \to p$ is also refuted in $\mathfrak{M}$, since $a \models (p \to \bot) \to \bot$ and $a \not\models p$.

Example 2.5 The formula $p \to ((p \to \bot) \to \bot)$ is valid in all intuitionistic frames. Indeed, suppose otherwise. Then there is a model on a frame $\mathfrak{F} = \langle W, R \rangle$ such that $x \models p$ and $x \not\models (p \to \bot) \to \bot$ for some $x \in W$, and so there is $y \in W$ for which xRy and $y \models p \to \bot$. By the definition of valuation, we must have $y \models p$, whence $y \not\models p \to \bot$, which is a contradiction.

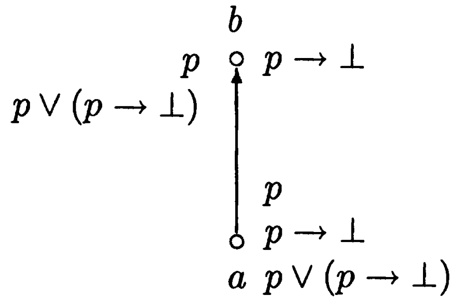

FIG. 2.1.

We define *intuitionistic propositional logic* $\mathbf{Int}_{\mathcal{L}}$ in the language $\mathcal{L}$ as the set of all $\mathcal{L}$-formulas that are valid in all intuitionistic frames, i.e.,

$$\mathbf{Int}_{\mathcal{L}} = \{\varphi \in \mathbf{For}\mathcal{L} : \ \mathfrak{F} \models \varphi \text{ for all frames } \mathfrak{F}\}.$$

Usually we will drop the subscript $\mathcal{L}$ and write simply **Int**.

Since the classical validity is nothing else but the validity in the single-point intuitionistic frame, we obtain the inclusion

$$\mathbf{Int} \subseteq \mathbf{Cl}.$$

And since $p \vee \neg p$ is in **Cl** but does not belong to **Int**, this inclusion is proper.

2.3 Truth-preserving operations

In comparison with classical models intuitionistic ones are much more complex structures. So before proceeding to the study of **Int** let us develop some notions and technical means for handling them. In this section we introduce three very important operations on intuitionistic models and frames which preserve truth and validity.

A frame $\mathfrak{G} = \langle V, S\rangle$ is called a *subframe* of a frame $\mathfrak{F} = \langle W, R\rangle$ (notation: $\mathfrak{G} \subseteq \mathfrak{F}$) if $V \subseteq W$ and S is the restriction of R to V ($S = R \restriction V$, in symbols), i.e., $S = R \cap V^2$. The subframe $\mathfrak{G}$ is a *generated subframe* of $\mathfrak{F}$ (notation: $\mathfrak{G} \subseteq_{\rightarrow} \mathfrak{F}$) if V is an upward closed subset of W.

Example 2.6 Let $\mathfrak{F}$ be the frame depicted in Fig. 2.2 (a). Then the frames shown in Fig. 2.2 (a)–(g) are (isomorphic to) subframes of $\mathfrak{F}$, with (a), (d), (e) and (f) being the only pairwise non-isomorphic generated subframes.

If $\mathfrak{G} = \langle V, S\rangle$ is a generated subframe of $\mathfrak{F} = \langle W, R\rangle$ and V is the upward closure of some set $X \subseteq W$, i.e., V is the minimal upward closed subset of W to contain X, then we say that V and $\mathfrak{G}$ are *generated by the set* X. Notice that since R is reflexive and transitive,

$$V = \{x \in W : \ \exists y \in X \ yRx\}.$$

If $\mathfrak{F}$ is generated by a singleton $\{x\}$ then $\mathfrak{F}$ is called *rooted* and x is called the *root* (or the *least point*) of $\mathfrak{F}$. All frames in Fig. 2.2, except (d) and (g), are rooted.

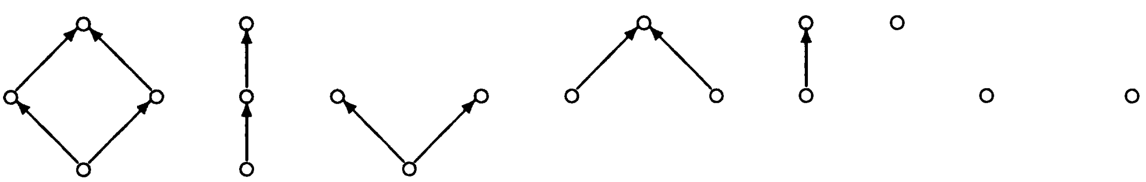

FIG. 2.2.

We introduce special notations for the operations of upward and downward closure. Namely, if $\mathfrak{F} = \langle W, R\rangle$ is a frame and $X \subseteq W$ then we let

$$X{\uparrow}R = \{x \in W : \exists y \in X\ yRx\},$$

$$X{\downarrow}R = \{x \in W : \exists y \in X\ xRy\}.$$

If $\mathfrak{F}$ is understood then we drop R and write simply $X{\uparrow}$ and $X{\downarrow}$; we also write $x{\uparrow}$ and $x{\downarrow}$ instead of $\{x\}{\uparrow}$ and $\{x\}{\downarrow}$, respectively. All the points in $x{\uparrow}$ ($x{\downarrow}$) are called *successors* (*predecessors*) of x; a successor (predecessor) y of x is *proper* if $x \neq y$. A proper successor (predecessor) y of x is an *immediate successor* (respectively, *immediate predecessor*) of x if $xRzRy$ ($yRzRx$) implies $z = x$ or $z = y$, for every $z \in W$. A point x is a *final* (or *maximal*) *point* in $\mathfrak{F}$ if $x{\uparrow} = \{x\}$; x is the *last* (or *greatest*) *point* in $\mathfrak{F}$ if $x{\downarrow} = W$. More generally, a point $x \in X \subseteq W$ is called *final* (or *maximal*) in X if no proper successor of x is in X.

Thus, $\mathfrak{G} = \langle V, S\rangle$ is a subframe of $\mathfrak{F} = \langle W, R\rangle$ generated by a set X if $V = X{\uparrow}R$ and $S = R \cap V^2$; x is the root of $\mathfrak{G}$ if $V = x{\uparrow}S$. Using arrows, instead of xRy we can write now either $y \in x{\uparrow}$ or $x \in y{\downarrow}$.

A model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$ is a *submodel* of a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ (notation: $\mathfrak{N} \subseteq \mathfrak{M}$) if $\mathfrak{G} = \langle V, S\rangle$ is a subframe of $\mathfrak{F} = \langle W, R\rangle$ and, for every $p \in \mathbf{Var}\mathcal{L}$,

$$\mathfrak{U}(p) = \mathfrak{V}(p) \cap V.$$

In the case when $\mathfrak{G} \subsetneq \mathfrak{F}$ the model $\mathfrak{N}$ is called a *generated submodel* of $\mathfrak{M}$ (notation: $\mathfrak{N} \subsetneq \mathfrak{M}$).

The formation of generated submodels is the first truth-preserving operation of the three mentioned above.

Theorem 2.7. (Generation) *Suppose $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U}\rangle$ is a generated submodel of $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$. Then for every formula φ and every point x in $\mathfrak{G}$,*

$$(\mathfrak{N}, x) \models \varphi \textit{ iff } (\mathfrak{M}, x) \models \varphi.$$

Proof The proof proceeds by induction on the construction of φ. The basis of induction is obvious. Let $\varphi = \psi \to \chi$, $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$. Then we have:

$$\begin{aligned}(\mathfrak{N}, x) \models \varphi &\text{ iff } \forall y \in x{\uparrow}S\ ((\mathfrak{N}, y) \models \psi \to (\mathfrak{N}, y) \models \chi)\\ &\text{ iff } \forall y \in x{\uparrow}R\ ((\mathfrak{M}, y) \models \psi \to (\mathfrak{M}, y) \models \chi)\\ &\text{ iff } (\mathfrak{M}, x) \models \varphi.\end{aligned}$$

Here the second equivalence is justified by the induction hypothesis and the fact that $x{\uparrow}S = x{\uparrow}R$, for every point $x \in V$.

The cases $\varphi = \psi \wedge \chi$ and $\varphi = \psi \vee \chi$ are trivial. ❑

The generation theorem means that the truth-values of formulas at a point x are completely determined by the truth-values of their variables at the points in $x\uparrow$ and do not depend on other points in the model.

Corollary 2.8 *If $\mathfrak{G} \subseteq \mathfrak{F}$ then, for every formula φ,*
(i) $(\mathfrak{G}, x) \models \varphi$ *iff* $(\mathfrak{F}, x) \models \varphi$, *for all points x in $\mathfrak{G}$;*
(ii) $\mathfrak{F} \models \varphi$ *implies* $\mathfrak{G} \models \varphi$.

Proof (i) Suppose $(\mathfrak{G}, x) \not\models \varphi$. Then $(\mathfrak{N}, x) \not\models \varphi$ for some model $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$. Define a valuation $\mathfrak{V}$ on $\mathfrak{F}$ by taking

$$\mathfrak{V}(p) = \mathfrak{U}(p) \text{ for all } p \in \mathbf{Var}\mathcal{L}.$$

Then $\mathfrak{N} \subseteq \mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ and so, by the generation theorem, $(\mathfrak{M}, x) \not\models \varphi$. Therefore, $(\mathfrak{F}, x) \models \varphi$ implies $(\mathfrak{G}, x) \models \varphi$. The converse implication is a direct consequence of the generation theorem.

(ii) follows from (i). ❑

We draw two more simple consequences of the generation theorem.

Corollary 2.9 *For every frame $\mathfrak{F}$ and every formula φ, the following conditions are equivalent:*
(i) $\mathfrak{F} \models \varphi$;
(ii) $\mathfrak{G} \models \varphi$, *for every* $\mathfrak{G} \subseteq \mathfrak{F}$;
(iii) $\mathfrak{G} \models \varphi$, *for every rooted* $\mathfrak{G} \subseteq \mathfrak{F}$.

Corollary 2.10 $\mathbf{Int}_{\mathcal{L}} = \{\varphi \in \mathbf{For}\mathcal{L} : \mathfrak{F} \models \varphi$ *for all rooted frames* $\mathfrak{F}\}$.

Our second truth-preserving operation is defined in a slightly more complicated way.

Suppose we have two frames $\mathfrak{F} = \langle W, R \rangle$ and $\mathfrak{G} = \langle V, S \rangle$. A map f from W *onto* V is called a *reduction of $\mathfrak{F}$ to $\mathfrak{G}$* if the following conditions hold for every $x, y \in W$:

(R1) xRy implies $f(x)Sf(y)$;

(R2) $f(x)Sf(y)$ implies $\exists z \in W\ (xRz \wedge f(z) = f(y))$.

In this case we say also that f *reduces $\mathfrak{F}$ to $\mathfrak{G}$* or $\mathfrak{G}$ is an *f-reduct* (or simply a *reduct*) of $\mathfrak{F}$ or $\mathfrak{F}$ is *f-reducible* (or simply *reducible*) to $\mathfrak{G}$. Such a map f is often called a *pseudo-epimorphism* or just a *p-morphism* as well.

Proposition 2.11 *A one-to-one reduction of $\mathfrak{F}$ to $\mathfrak{G}$ is an isomorphism between $\mathfrak{F}$ and $\mathfrak{G}$.*

Proof Exercise. ❑