Intuitionistic and Modal Logic

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Overview

- Introduction to intuitionism
- Proofs and proof systems (IPC)
- Kripke models, Metatheorems
- Modal Logic, Translations
- Formulas of one variable
- Heyting algebras
- Universal models for IPC, Jankov formulas, intermediate logics
- Universal models for modal logic, application to unifiablity

Introduction, Brouwer

- First three lectures partly based on Intuitionistic Logic by Nick Bezhanishvili and Dick de Jongh, Lecture Notes for ESSLI 2005, ILLC-Prepublications PP-2006-25, www.illc.uva.nl
- See also Nick Bezhanishvili, Lattices of Intermediate and Cylindric Modal Logics, Dissertation, Universiteit van Amsterdam, ILLC Dissertation Series DS-2006-02
- A. Chagrov and M. Zakharyaschev, Modal Logic, Oxford University Press, 1997.
- Brouwer (1881-1963)
- Before Brouwer, Foundations of analysis (Cauchy, Weierstrass), non-euclidean geometry,

- Dissertation 1907: Over de Grondslagen van de Wiskunde (About the Foundations of Mathematics).

Foundations of mathematics

- Period that Foundations of Mathematics was hot issue.
- Frege 1879 Begriffschrift, 1884 Grundlagen der Mathematik, 1903 Grundgesetze der Arithmetik,
- 1901: Russell's paradox, Russell 1903, Principles of Mathematics,
- Hilbert 1889 Grundlagen der Geometrie, 1900 Mathematische Probleme, 1904 Über die Grundlagen der Logik und der Arithmetik,
- Cantor . . ., Peano, Schröder, Huntington, Veblen, . . .

Precursors of Intuitionism

- Predecessors of Brouwer: Kronecker: "God made the natural numbers, the rest is human work", French semi-intuitionists (e.g. E. Borel).

- Unhappy feeling from these mathematicians about abstractness of mathematics, proving the existence of objects by reasoning by contradiction, so that no object really arises from the proof: $\neg\forall x \neg Ax \rightarrow \exists x\, Ax$.

Intuitionism, Platonism, Formalism

- Intuitionism is as one of the three basic points of view opposed to Platonism and formalism. View that mathematics and mathematical truths are creations of the human mind: true = provable. N.B! provable in the informal, not formal sense.
- Platonism. Most famous modern representatives: Frege, Gödel. View that mathematical objects have independent existence outside of space-time, that mathematical truths are independent of us. At the time mixed with logicism, Frege's idea that mathematics is no more than logic, since mathematics can be reduced to it, a view supported by Russell (not a Platonist) at the time.
- Formalism. Most famous modern representative: Hilbert. View that there are no mathematical objects, no mathematical truths, just formal systems and derivations in them.

Brouwer's ideas

- Foundations unnecessary, in fact impossible,
- Logic follows mathematics, is not its basis, logical rules extracted from mathematics,
- Mathematics is a mental activity, the "exact part of human thought", writing mathematics down is only an aide,
- Criticism of 'classical' logical laws,
- Principle of the excluded third (law of the exclude middle) $A \vee \neg A$.

Brouwer's programme

- Brouwer's programme: rebuilding of mathematics according to intuitionistic principles.
- Only partially successful. Not accepted by mathematicians in practice.
- But study of intuitionistic proofs and formal systems very alive. Only by fully accepting intuitionistic methods does one get proofs that guarantees to exhibit objects that are proved to exist. One gets this way the constructive part of mathematics.
- Less popular but fascinating are Brouwer's choice sequences which have classically inconsistent properties.
- Our course will mostly concentrate on propositional calculus.

Example of nonconstructive proof

- Theorem There exist irrational numbers r and s such that r^s is rational.
- Proof Well-known since Euclid, $\sqrt{2}$ is irrational.
- Now either $\sqrt{2}^{\sqrt{2}}$ is rational or it is not.
- In the first case take $r=\sqrt{2}$, $s=\sqrt{2}$. Then $r^s=2$, i.e. rational.
- In the second, take $r=\sqrt{2}^{\sqrt{2}}$, $s=\sqrt{2}$. Then $r^s=(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}=\sqrt{2}^2=2$, i.e. rational.
- So, we have found r and s as required, only we cannot tell what r is, it is either $\sqrt{2}^{\sqrt{2}}$ or $\sqrt{2}$ (in reality of course the latter) and $=\sqrt{2}$.

Heyting

- Heyting, 1928-1930:
- Earlier incomplete version in Kolmogorov 1925,
- Hilbert type system. We first give natural deduction variant of which first version was given by Gentzen.
- $\neg\varphi$ is defined as $\varphi \to \bot$ where $\bot$ stands for a contradiction, an obviously false statement like $1 = 0$.

Natural Deduction

	introduction	elimination
$\bot$	none	$\dfrac{\bot}{\varphi}$
$\to$	$[\varphi]$ $\vdots$ $\dfrac{\psi}{\varphi \to \psi}$	$\dfrac{\varphi \quad \varphi \to \psi}{\psi}$
$\wedge$	$\dfrac{\varphi \quad \psi}{\varphi \wedge \psi}$	$\dfrac{\varphi \wedge \psi}{\varphi} \quad \dfrac{\varphi \wedge \psi}{\psi}$
$\vee$	$\dfrac{\varphi}{\varphi \vee \psi} \quad \dfrac{\psi}{\varphi \vee \psi}$	$\varphi \quad \psi$ $\vdots \quad \vdots$ $\dfrac{\varphi \vee \psi \quad \chi \quad \chi}{\chi}$
$\forall$	$\dfrac{\varphi(x)}{\forall x\, \varphi(x)}$	$\dfrac{\forall x\, \varphi(x)}{\varphi(t)}$
$\exists$	$\dfrac{E(t)}{\exists x\, E(x)}$	$E(c)$ $\vdots$ $\dfrac{\exists x\, E(x) \quad P}{P}$

Classical Logic

To get classical logic one adds the rule that if $\bot$ is derived from $\neg\varphi$, then one can conclude to φ dropping the assumption $\neg\varphi$.

$$\varphi \to \bot$$

$$\vdots$$

$$\frac{\bot}{\neg\varphi}$$

BHK-interpretation

- Brouwer-Heyting-Kolmogorov Interpretation of connectives and quantifiers.

  Natural deduction closely related to BHK.

- Interpretation by means of proofs (nonformal, nonsyntactical objects, mind constructions),
- A proof of $\varphi \wedge \psi$ consists of proof of φ plus proof of ψ (plus conclusion),
- A proof of $\varphi \vee \psi$ consists of proof of φ or of proof of ψ (plus conclusion),
- A proof of $\varphi \to \psi$ consists of method that applied to any conceivable proof of φ will deliver proof of ψ,

BHK-interpretation, continued

- Nothing is a proof of $\bot$,
- Proof of $\neg\varphi$ is method that given any proof of φ gives proof of $\bot$,
- A proof of $\exists x\, \varphi(x)$ consists of object d from domain plus proof of $\varphi(d)$ (plus conclusion),
- A proof of $\forall x\, \varphi(x)$ consists of method that applied to any element d of domain will deliver proof of $\varphi(d)$,

Valid and invalid reasoning

- A disjunction is hard to prove: e.g. of the four directions of the de Morgan laws only $\neg(\varphi \wedge \psi) \rightarrow \neg\varphi \vee \neg\psi$ is not valid,
- $\neg(\varphi \vee \psi) \rightarrow \neg\varphi \wedge \neg\psi$,
- $(\neg\varphi \wedge \neg\psi) \rightarrow \neg(\varphi \vee \psi)$,
- $\neg\varphi \vee \neg\psi \rightarrow \neg(\varphi \wedge \psi)$ are valid,
- other examples of such invalid formulas are $\varphi \vee \neg\varphi$, (the law of the the excluded middle)
- $\neg(\varphi \wedge \psi) \rightarrow \neg\varphi \vee \neg\psi$,
- $(\varphi \rightarrow \psi \vee \chi) \rightarrow (\varphi \rightarrow \psi) \vee (\varphi \rightarrow \chi)$,
- $((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow \varphi \vee \psi$,

Valid and invalid reasoning, continued

- An existential statement is hard to prove:
- of the four directions of the classically valid interactions between negations and quantifiers only $\neg\forall x\,\varphi \rightarrow \exists x \neg\varphi$ is not valid,
- statements directly based on the two-valuednes of truth values are not valid, e.g. $\neg\neg\varphi \rightarrow \varphi$ or $((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$ ($Peirce's\ law$),
- On the other hand, many basic laws naturally remain valid, commutativity and associativity of conjunction and disjunction, both distributivity laws,
- $(\varphi \rightarrow \psi \wedge \chi) \leftrightarrow (\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)$,
- $(\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi) \leftrightarrow (\varphi \vee \psi \rightarrow \chi))$,
- $(\varphi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow (\varphi \wedge \psi) \rightarrow \chi$,
- $((\varphi \vee \psi) \wedge \neg\varphi \rightarrow \psi))$ (needs $ex\ falso$!).

Hilbert type system

- $\varphi \to (\psi \to \varphi)$
- $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)))$
- The only rule is modus ponens from φ and $\varphi \to \psi$ conclude ψ.
- The first two axioms plus modus ponens are sufficient for proving the deduction theorem. (corresponding to implication introduction).
- $\varphi \wedge \psi \to \varphi \quad \varphi \wedge \psi \to \psi$,
- $\varphi \to (\psi \to \varphi \wedge \psi)$,
- $\varphi \to \varphi \vee \psi \quad \psi \to \varphi \vee \psi$,
- $(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$,
- $\bot \to \varphi$,

Classical propositional calculus

- To get CPC add $((\varphi \to \psi) \to \varphi) \to \varphi$ (Peirce's law) or $\neg\neg\varphi \to \varphi$.

Kripke frames and models

- Frames, (usually $\mathfrak{F}$):
- A set of worlds W, also nodes, points
- An accessibility relation R, which is a $\leqslant$-partial order,
- For models $\mathfrak{M}$ a persistent valuation V is added. Persistence means:
- $wRw' \,\&\, w \in V(p) \Longrightarrow w \in V'(p)$.
- $w \models \varphi \wedge \psi \Longleftrightarrow w \models \varphi$ and $w \models \psi$,
- $w \models \varphi \vee \psi \Longleftrightarrow w \models \varphi$ or $w \models \psi$,
- $w \models \varphi \rightarrow \psi \Longleftrightarrow \forall w'(wRw'$ and $w' \models \varphi \Rightarrow w' \models \psi)$,

Kripke frames and models, continued

- Frames will usually have a root w_0: $w_0 \, R \, w$ for all w.
- $w \not\models \bot$,
- $w \models \neg\varphi \iff \forall w'(wRw' \Rightarrow$ not $w \models \varphi)$ (follows from definition of $\neg\varphi$ as $\varphi \to \bot$),
- Persistence for formulas follows:
- $wRw' \,\&\, w \models \varphi \implies w' \models \varphi$.
- Note that $w \models \neg\neg\varphi \iff \forall w'(wRw' \implies \exists w''(w'Rw'' \,\&\, w'' \models \varphi))$
- $\Leftrightarrow$ for finite models $\leftrightarrow \forall w''(wRw'' \,\&\, w''$ end point $\implies w'' \models \varphi)$.

Kripke frames and models, predicate logic

- Increasing domains D_w:
- $wRw' \Longrightarrow D_w \subseteq D_{w'}$.
- with names for the elements of the domains:
- $w \models \exists x\varphi(x) \iff$, for some $d \in D_w$, $w \models \varphi(d)$,
- $w \models \forall x\varphi(x) \iff$, for each w' with wRw' and all $d \in D_{w'}$, $w' \models \varphi(d)$,
- Persistency transfers to formulas here as well.

Counter-models to propositional formulas

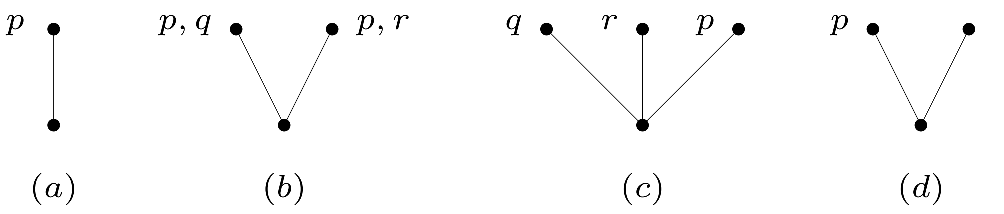


Figure 1: Counter-models for the propositional formulas

- These figures give counterexamples to respectively:
- (a) $p \vee \neg p$, $\neg\neg p \rightarrow p$,
- (b) $(p \rightarrow q \vee r) \rightarrow (p \rightarrow q) \vee (p \rightarrow r)$,
- (c) $(\neg p \rightarrow q \vee r) \rightarrow (\neg p \rightarrow q) \vee (p \rightarrow r)$,
- (d) $(\neg\neg p \rightarrow p) \rightarrow p \vee \neg p$.

Counter-models to predicate formulas

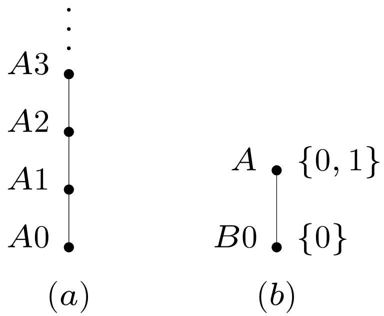


Figure 2: Counter-models for the predicate formulas

- These figures give counterexamples to:
- (a) $\neg\neg\forall x(Ax \vee \neg Ax)$, if domain constant $\mathbb{N}$ (and also against $\forall x\neg\neg Ax \rightarrow \neg\neg\forall x Ax$),
- (b) $\forall x(A \vee Bx) \rightarrow A \vee \forall x Bx$.

Soundness and Completeness

- φ is valid in a model $\mathfrak{M}$, $\mathfrak{M} \models \varphi$, if φ is satisfied in all worlds in the model. φ is valid in a frame $mathfrakF$, $mathfrakF \models \varphi$ if φ is valid in all models on the frame.
- Completeness Theorem:
- $\vdash_{\mathrm{IPC}} \varphi$ iff φ is valid in all (finite) frames.
- Soundness ($\Longrightarrow$) just means checking all axioms in Hilbert type system (plus the fact that modus ponens leaves validity intact).

Glivenko's theorem

- Before the completeness proof an application of completeness.
- Glivenko's Theorem, Theorem 5:
- $\vdash_{\mathrm{CPC}} \varphi$ iff $\vdash_{\mathrm{IPC}} \neg\neg\varphi$ (CPC is classical propositional calculus).
- $\Longleftarrow$ is of course trivial.
- $\Longrightarrow$ Exercise.
- e.g. $\vdash_{\mathrm{IPC}} \neg\neg(\varphi \vee \neg\varphi)$.
- Glivenko's Theorem does not extend to predicate logic, exercise.

Proof of Completeness

- Basic entities in Henkin type completeness proof are:
- Theories with the disjunction property,
- A set Γ of formulas is a theory if Γ is closed under IPC-induction.
- A set Γ of formulas has the disjunction property if $\varphi \vee \psi \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$.
- Lindenbaum type lemma needed.

Lemma

- Lemma 10 If $\Gamma \cup \{\psi\} \nvdash_{\text{IPC}} \chi$, then a theory with the disjunction property Δ exists such that $\Gamma \subseteq \Delta$, $\psi \in \Delta$ and $\chi \notin \Delta$.

- Proof.

  Enumerate all formulas: $\varphi_0, \varphi_1, \cdots$ and define:

- $\Delta_0 = \Gamma \cup \{\psi\}$,

- $\Delta_{n+1} = \Delta_n \cup \{\varphi_n\}$ if this does not prove χ,

- $\Delta_{n+1} = \Delta_n$ otherwise.

- Δ is the union of all Δ_n.

- $\Delta_n \nvdash_{\text{IPC}} \chi$, $\Delta \nvdash_{\text{IPC}} \chi$,

Proof lemma, continuation

- Δ is a theory.
- Claim: Δ has the disjunction property:
- Assume $\varphi \vee \psi \in \Delta$, $\varphi \notin \Delta$, $\theta \notin \Delta$.
- Let $\varphi = \varphi_m$ and $\theta = \varphi_n$ and w.l.o.g. let $n \geqslant m$.
- $\Delta_n \cup \{\varphi\} \vdash_{\mathrm{IPC}} \chi$ and $\Delta_n \cup \{\theta\} \vdash_{\mathrm{IPC}} \chi$, and thus $\Delta_n \cup \{\varphi \vee \theta\} \vdash_{\mathrm{IPC}} \chi$. But $\Delta_n \cup \{\varphi \vee \theta\} \subseteq \Delta$ and $\Delta \nvdash_{\mathbf{IPC}} \chi$, Contradiction.

Canonical model

- $\mathfrak{M}^C$
- W^C is the set of all consistent theories with the disjunction property,
- $R^C = \subseteq$,
- Frame of canonical model is $\mathfrak{F}^C = (W^C, R^C)$.
- Valuation of V^C of canonical model: $\Gamma \in V^C(p) \Leftrightarrow \Gamma \models p \Leftrightarrow p \in \Gamma$.
- The construction can be restricted to formulas in n variables. We then get the n-canonical model (or n-Henkin model.

Completeness of IPC

- Theorem 12. $\Gamma \vdash_{\mathrm{IPC}} \varphi$ iff φ is valid in all Kripke models of Γ for IPC.
- For the Completeness side $(\Leftarrow)$ we show: if $\Gamma \nvdash_{\mathrm{IPC}} \varphi$, then $\varphi \notin \Delta$ for some Δ containing Γ in the canonical model.
- First show by induction on ψ that $\Theta \models \psi \Leftrightarrow \psi \in \Theta$.
- Most cases easy: it is for example necessary to show that $\psi \wedge \chi \in \Theta \Leftrightarrow \psi \in \Theta \,\&\, \chi \in \Theta$. This follows immediately from the fact that Θ is a theory (closed under IPC-induction). The corresponding fact for $\vee$ is the disjunction property.

Completeness of IPC, continued

- The hardest is showing that, if $\psi \to \chi \notin \Theta$, then a theory Δ with the disjunction property such that $\Theta \subseteq \Delta$ exists with $\psi \in \Delta$ and $\chi \notin \Delta$.
- But this is the content of Lemma 10.
- Now assume $\Gamma \nvdash_{\mathrm{IPC}} \varphi$. Then $\Gamma \nvdash_{\mathrm{IPC}} \top \to \varphi$. Lemma 10 supplies the required Δ.

Finite Model Property

- Theorem For finite Γ, $\Gamma \vdash_{\mathrm{IPC}} \varphi$ iff φ is valid in all finite Kripke models of Γ for IPC.

- Proof. The proof can be done by filtration. We will not do that here. Or by reducing the whole discussion to the set of subformulas of $\Gamma \cup \{\varphi\}$ (a so-called adequate set, both in the definition of the (reduced) canonical model as well as in the proof.

- Same for a language with only finitely many propositional variables. (Model will not be finite!)

Completeness of Predicate Logic

- Let $C_0, C_1, C_2, \cdots$ be a sequence of disjoint countably infinite sets of new constants. It suffices to consider theories in the languages $\mathcal{L}_n$ obtained by adding $C_0 \cup C_1 \cdots \cup C_n$ to the original language $\mathcal{L}$. We consider theories containing $\exists x\varphi(x) \to \varphi(c_\varphi)$ as in the classical Henkin proof. That will immediately guarantee that the theories besides the disjunction property, also have the analogous existence property. The proof then proceeds as in the propositional case. The role of the additional constants becomes clear in the induction step for the universal quantifier:
- If Θ is a theory in $\mathcal{L}_n$. To show is:
- $\forall x\, \varphi(x) \in \Theta$ iff, for each d and Θ' in $\mathcal{L}_m$ (m $\geqslant$ n) with $\Theta \subseteq \Theta'$, $\varphi(d) \in \Theta'$.
- $\Rightarrow$ is of course obvious because Θ' is a theory.

Completeness of Predicate Logic, continued

For $\Leftarrow$ assume that $\forall x\, \varphi(x) \notin \Theta$. Then, for some new constant d in C_{n+1}, $\Theta \nvdash \varphi(d)$. And hence Θ can be extended to a Henkin theory Θ' with the disjunction property in $\mathcal{L}_{n+1}$ that does not prove $\varphi(d)$ either.

Generated subframes and submodels, disjoint unions

- Definition 7. $R(w) = \{w' \in W \mid wRw'\}$,
- The generated subframe $\mathfrak{F}_w$ of $\mathfrak{F}$ is $(R(w), R')$, where R' the restriction of R to $R(w)$.
- The generated submodel $\mathfrak{K}_w$ of $\mathfrak{K}$ is $\mathfrak{F}_w$ with V restricted to it.
- If $\mathfrak{F}_1 = (W_1, R_1)$ and $\mathfrak{F}_2 = (W_2, R_2)$, then their disjoint union $\mathfrak{F}_1 \uplus \mathfrak{F}_2$ has as its set of worlds the disjoint union of W_1 and W_2, and R is $R_1 \cup R_2$. To get the disjoint union of two models the union of the two valuations is added.

p-morphisms

- If $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$ are frames, then f: $W \to W'$ is a p-morphism (also bounded morphism) from $\mathfrak{F}$ to $\mathfrak{F}'$ iff
- for each $w, w' \in W$, if wRw', then $f(w)Rf(w')$,
- for each $w \in W$, $w' \in W'$, if $f(w)Rw'$, then there exists $w'' \in W$, wRw'' and $f(w'') = w'$.
- If $\mathfrak{K} = (W, R, V)$ and $\mathfrak{K}' = (W', R', V')$ are models, then f: $W \to W'$ is a p-morphism from $\mathfrak{K}$ to $\mathfrak{K}'$ iff f is a p-morphism of the frames and, for all $w \in W$, $w \in V(p)$ iff $f(w) \in V'(p)$.

Properties of Generated Subframes

- Lemma
- If w' in the generated submodel $\mathfrak{M}_w$, then, $w' \models \varphi$ in $\mathfrak{M}$ iff $w' \models \varphi$ in $\mathfrak{M}_w$.
- This implies that if φ is falsified in a model, we may w.l.o.g. assume that it is falsified in the root.
- If $\mathfrak{F} \models \varphi$, then $\mathfrak{F}_w \models \varphi$.

Properties of p-morphic images, disjoint unions

- If f is a p-morphism from $\mathfrak{M}$ to $\mathfrak{M}'$ and $w \in W$, then $w \models \varphi$ iff $f(w) \models \varphi$.
- If $\mathfrak{F} \models \varphi$, then $\mathfrak{F}_w \models \varphi$.
- If f is a p-morphism from $\mathfrak{F}$ onto $\mathfrak{F}'$, then $\mathfrak{F} \models \varphi$ implies $\mathfrak{F}' \models \varphi$.
- If $w \in W_1$, then $w \models \varphi$ in $\mathfrak{M}_1 \uplus \mathfrak{M}_2$ iff $w \models \varphi$ in $\mathfrak{M}_1$, etc.

Disjunction property

- Theorem 16. $\vdash_{\mathrm{IPC}} \varphi \vee \psi$ iff $\vdash_{\mathrm{IPC}} \varphi$ or $\vdash_{\mathrm{IPC}} \psi$.
- This extends to the predicate calculus and arithmetic.
- Proof. $\Leftarrow$: Trivial

  $\Rightarrow$: Assume $\nvdash_{\mathrm{IPC}} \varphi$ and $\nvdash_{\mathrm{IPC}} \psi$.
- Let $\mathfrak{K} \not\models \varphi$ and $\mathfrak{L} \not\models \psi$.
- Add a new root w_0 below both $\mathfrak{K}$ and $\mathfrak{L}$. In w_0, $\varphi \vee \psi$ is falsified (because of persistence!).

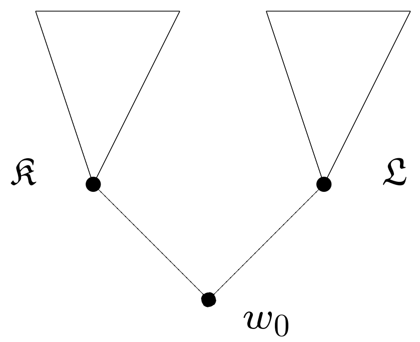


Figure 3: Proving the disjunction property

Modal Logic

- The language of modal logic is the language of the propositional calculus with an additional 1-place operator $\Box$ (pronounced: necessary),
- The basic modal logic **K** has as in addition to the axiom schemes of the calssical propositional calculus **CPC** the axiom scheme

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

  and the rule of necessitation $\varphi/\Box\varphi$

- An often used theorem is $\Box\varphi \wedge \Box\psi \leftrightarrow \Box(\varphi \wedge \psi)$.

S4, Grz and GL

- The modal-logical systems S4, Grz and GL are obtained by adding to
- The axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ of K,
- The axioms $\Box\varphi \to \varphi$, $\Box\varphi \to \Box\Box\varphi$ for S4
- In addition to this Grzegorczyk's axiom $\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$ for Grz,
- and $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ for GL.

Kripke frames and models for K

- Frames:
- A set of worlds W, also nodes, points
- An accessibility relation R,
- For models a valuation V is added.
- $wRw' \,\&\, w \in V(p) \Longrightarrow w \in V'(p)$.
- $w \models \varphi \wedge \psi \Longleftrightarrow w \models \varphi$ and $w \models \psi$, etc.
- $w \models \Box\varphi \Longleftrightarrow \forall w'(wRw' \Rightarrow w' \models \varphi)$,

Completeness of K

- Basic entities in Henkin type completeness proof for **K** are:
- Maximal consistent sets (these are of course also theories with the disjunction property),
- lemma needed.
- Lemma If $\{\Box\varphi \mid \varphi \in \Gamma\} \nvdash_K \Box\psi$, then $\Gamma \nvdash \psi$,

  Proof If $\Gamma \vdash \psi$, then $\{\Box\varphi \mid \varphi \in \Gamma\} \vdash_K \Box\psi$

Canonical model of K

- The canonical model $\mathfrak{M}^{\mathbf{K}}$ is defined as follows:
- $\mathfrak{M}^{\mathbf{K}} = (W^{\mathbf{K}}, W^{\mathbf{K}}, V^{\mathbf{K}}) = (\mathfrak{F}^{\mathbf{K}}, V^{\mathbf{K}})$
- $W^{\mathbf{K}}$ is the set of all maximal consistent sets,
- $\Gamma R^{\mathbf{K}} \Delta \leftrightarrow (\forall \Box\varphi \in \Gamma \Rightarrow \varphi \in \Delta)$,
- Frame of canonical model is $\mathfrak{F}^{\mathbf{K}} = (W^{\mathbf{K}}, R^{\mathbf{K}})$.
- Valuation of $V^{\mathbf{K}}$ of canonical model: $\Gamma \in V^{\mathbf{K}}(p) \Leftrightarrow \Gamma \models p \Leftrightarrow p \in \Gamma$.

Validity on models, frames, characterization

- Definition

$$\mathfrak{M} \models \varphi \Leftrightarrow \forall w \in W (w \models \varphi)$$

$$\mathfrak{F} \models \varphi \Leftrightarrow \forall \mathfrak{M} \,\mathrm{on}\, \mathfrak{F} (\mathfrak{M} \models \varphi)$$

- A modal logic L is said to define or characterize the class of frames $\mathfrak{F}$ such that $\mathfrak{F} \models L$.

Kripke frames, models for S4, Grz and GL

- S4 characterizes the reflexive transitive frames,
- S4 is complete w.r.t. the (finite) reflexive, transitive frames,
- S4 is complete w.r.t. $\leqslant$-partial orders (reflexive, transitive, anti-symmetric)
- Grz characterizes the reflexive, transitive, conversely well-founded frames,
- Grz is complete w.r.t. the finite $\leqslant$-partial orders,
- GL characterizes the transitive, conversely well-founded (i.e. irreflexive, asymmetric) frames.
- GL is complete w.r.t. the finite $<$-partial orders.

Translations

- Gödel's negative translation
- extends to the predicate calculus and arithmetic, has many variations,
- Definition 28
- $p^n = \neg\neg p$,
- $(\varphi \wedge \psi)^n = \varphi^n \wedge \psi^n$,
- $(\varphi \vee \psi)^n = \neg\neg(\varphi^n \vee \psi^n)$,
- $(\varphi \to \psi)^n = \varphi^n \to \psi^n$,
- $\bot^n = \bot$.

Properties of Gödel's negative translation

- Theorem 29. $\vdash_{\mathrm{CPC}} \varphi$ iff $\vdash_{\mathrm{IPC}} \varphi^n$.
- Proof.
- $\Longleftarrow$: $\vdash_{\mathrm{IPC}} \varphi^n \Rightarrow \vdash_{\mathrm{CPC}} \varphi^n \Rightarrow \vdash_{\mathrm{CPC}} \varphi$.

  $\Longrightarrow$: First prove $\vdash_{\mathrm{IPC}} \varphi^n \leftrightarrow \neg\neg\varphi^n$ (φ^n is negative) (using $\vdash_{\mathrm{IPC}} \neg\neg(\varphi \to \psi) \leftrightarrow (\neg\neg\varphi \to \neg\neg\psi)$ and $\vdash_{\mathrm{IPC}} \neg\neg(\varphi \wedge \psi) \leftrightarrow (\neg\neg\varphi \wedge \neg\neg\psi)$. Then simply follow the proof of φ in CPC to mimic it with a proof of φ^n in IPC. Exercise.

Gödel's translation of IPC into S4

- Gödel noticed the closeness of S4 and IPC when one interprets $\Box$ as intuitive provability.
- Definition 32.
- $p^\Box = \Box p$,
- $(\varphi \wedge \psi)^\Box = \varphi^\Box \wedge \psi^\Box$,
- $(\varphi \vee \psi)^\Box = \varphi^\Box \vee \psi^\Box$,
- $(\varphi \to \psi)^\Box = \Box(\varphi^\Box \to \psi^\Box)$,
- Theorem 33 $\vdash_{\mathrm{IPC}} \varphi$ iff $\vdash_{\mathrm{S4}} \varphi^\Box$ iff $\vdash_{\mathrm{Grz}} \varphi^\Box$.

Proof for Gödel's translation of IPC into S4

- Proof $\Longrightarrow$: Trivial from S4 to Grz. From IPC to S4 it is simply a matter of using one of the proof systems of IPC and to find the needed proofs in S4, or showing their validity in the S4-frames and using completeness.

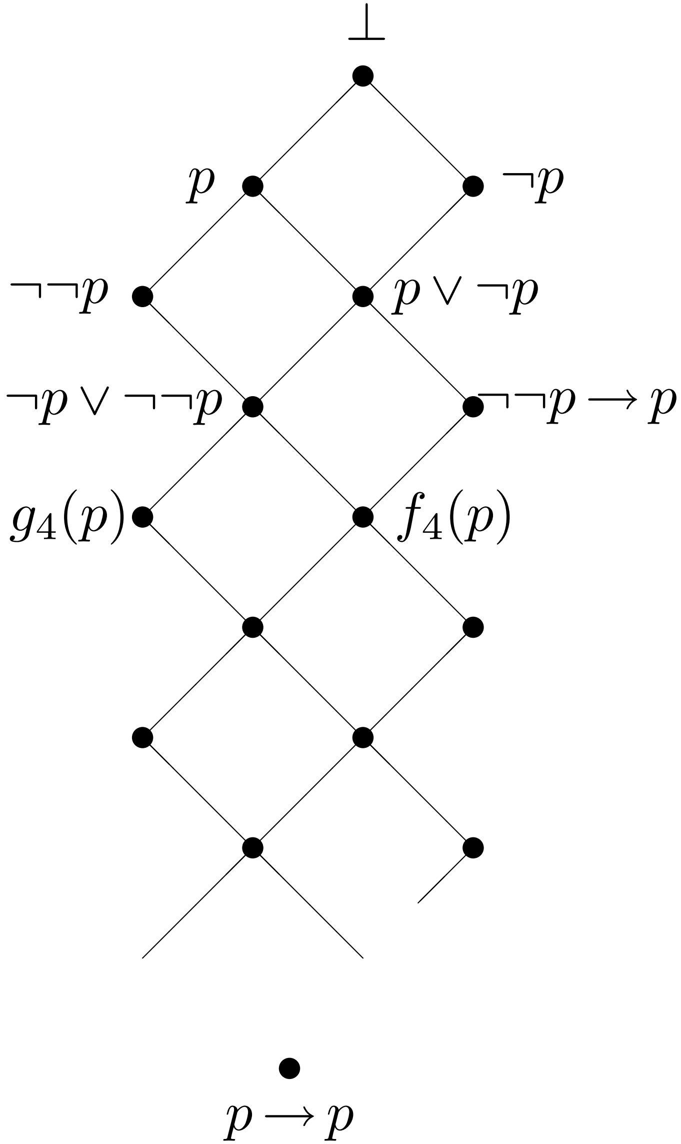
- $\Longleftarrow$: It is sufficient to note that it is easily provable by induction on the length of the formula φ that for any world w in a Kripke model with a persistent valuation $w \models \varphi$ iff $w \models \varphi^{\Box}$. This means that if $\nvdash_{\mathrm{IPC}} \varphi$ one can interpret the finite IPC-countermodel to φ provided by the completeness theorem immediately as a finite Grz-countermodel to $\varphi^{\Box}$.

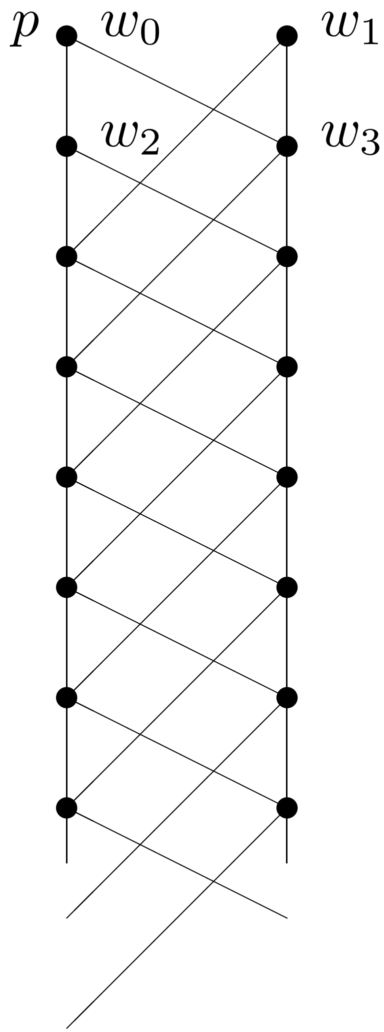
Intermediate Logics

- Intermediate logics (Superintuitionistic logics),
- Logics extending intuitionistic logic by axiom schemes (and sublogics of classical logic),
- e.g. Weak excluded middle: $\neg\varphi \vee \neg\neg\varphi$,
- Dummett's logic: $(\varphi \to \psi) \vee (\psi \to \varphi)$,
- most do not have disjunction property, some do:
- e.g. the Kreisel-Putnam logic $(\neg\varphi \to \psi \vee \chi) \to (\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)$,

The Rieger-Nishimura Lattice and Ladder

- Definition 36. Rieger-Nishimura Lattice.
- $g_0(\varphi) = f_0(\varphi) =_{def} \varphi$,
- $g_1(\varphi) = f_1(\varphi) =_{def} \neg\varphi$,
- $g_2(\varphi) =_{def} \neg\neg\varphi$,
- $g_3(\varphi) =_{def} \neg\neg\varphi \rightarrow \varphi$,
- $g_{n+4}(\varphi) =_{def} g_{n+3}(\varphi) \rightarrow g_n(\varphi) \vee g_{n+1}(\varphi)$,
- $f_{n+2}(\varphi) =_{def} g_n(\varphi) \vee g_{n+1}(\varphi)$.


⊥
p
$\neg p$
$\neg\neg p$
$p \vee \neg p$
$\neg p \vee \neg\neg p$
$\neg\neg p \rightarrow p$
$g_4(p)$
$f_4(p)$
$p \rightarrow p$


p
w_0
w_1
w_2
w_3

The Rieger-Nishimura Lattice and Ladder II

- Theorem 37.
- Each formula $\varphi(p)$ with only the propositional variable p is IPC-equivalent to a formula $f_n(p)$ $(n \geqslant 2)$ or $g_n(p)$ $(n \geqslant 0)$, or to $\top$ or $\bot$.
- All formulas $f_n(p)$ $(n \geqslant 2)$and $g_n(p)$ $(n \geqslant 0)$ are nonequivalent in IPC. In fact, in the Rieger-Nishimura Ladder w_i validates $g_n(p)$ for $i \geqslant n$ only.
- In the Rieger-Nishimura lattice a formula $\varphi(p)$ implies $\psi(p)$ in IPC iff $\psi(p)$ can be reached from $\varphi(p)$ by a downward going line.
- The frame of the Rieger-Nishimura ladder will be called $\mathcal{RN}$. Its subframes generated by w_k will be called $\mathcal{RN}_k$.

Heyting algebras

Overview

- Lattices, distributive lattices and Heyting algebras
- Heyting algebras and Kripke frames
- Algebraic completeness of **IPC**

Lattices

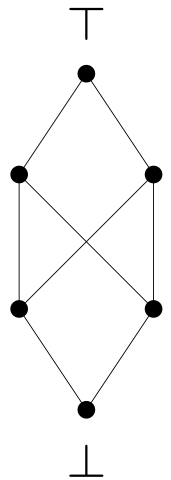
A partially ordered set $(A, \leq)$ is called a lattice if every two element subset of A has a least upper and greatest lower bound.

Let $(A, \leq)$ be a lattice. For $a, b \in A$ let

$a \vee b := sup\{a, b\}$ and $a \wedge b := inf\{a, b\}$.

Lattices, top and bottom

We assume that every lattice is bounded, i.e., it has a least and a greatest element denoted by $\bot$ and $\top$ respectively.



Lattices, axioms

Proposition 40. A structure $(A, \vee, \wedge, \bot, \top)$ is a lattice iff for every $a, b, c \in A$ the following holds:

1. $a \vee a = a$, $a \wedge a = a$; (idempotency laws)
2. $a \vee b = b \vee a$, $a \wedge b = b \wedge a$; (commutative laws)
3. $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$; (associative laws)
4. $a \vee \bot = a$, $a \wedge \top = a$; (existence of $\bot$ and $\top$)
5. $a \vee (b \wedge a) = a$, $a \wedge (b \vee a) = a$. (absorption laws)

Lattices, axioms, continued

Proof.(Sketch)

$\Rightarrow$ Check that every lattice satisfies the axioms 1–5.

$\Leftarrow$ Suppose $(A, \vee, \wedge, \bot, \top)$ satisfies the axioms 1–5.

Define $a \leq b$ by putting $a \vee b = b$ or equivalently by putting $a \wedge b = a$.

Check that $(A, \leq)$ is a lattice. $\square$

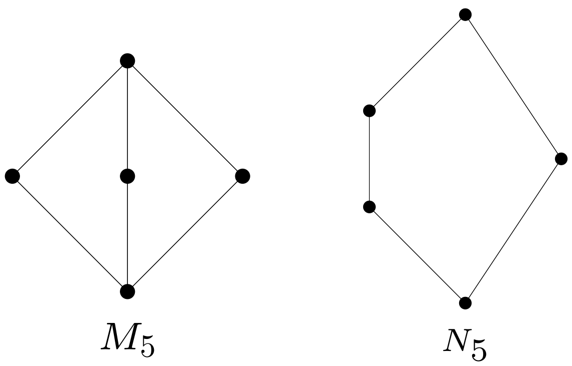
We denote lattices by $(A, \vee, \wedge, \bot, \top)$.

Distributive lattices

Definition 41. A lattice $(A, \vee, \wedge, \bot, \top)$ is called distributive if it satisfies the distributive laws:

- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

The lattices M_5 and N_5 are not distributive.



Distributive lattices, characterization

Theorem 43. A lattice L is distributive iff M_5 and N_5 are not sublattices of L.

Heyting algebras

Definition 44. A distributive lattice $(A, \wedge, \vee, \bot, \top)$ is said to be a Heyting algebra if for every $a, b \in A$ there exists an element $a \to b$ such that for every $c \in A$ we have:

$$c \leq a \to b \text{ iff } a \wedge c \leq b.$$

In every Heyting algebra $\mathfrak{A}$ we have that

$$a \to b = \bigvee \{c \in A : a \wedge c \leq b\}.$$

Heying algebras, axioms

Theorem 47. A (distributive) lattice $\mathfrak{A} = (A, \wedge, \vee, \bot, \top)$ is a Heyting algebra iff there is a binary operation $\to$ on A such that for every $a, b, c \in A$:

1. $a \to a = \top$
2. $a \wedge (a \to b) = a \wedge b$
3. $b \wedge (a \to b) = b$
4. $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$

Complete distributive lattices

We say that a lattice $(A, \wedge, \vee)$ is complete if for every subset $X \subset A$ there exist $inf(X) := \bigwedge X$ and $sup(X) := \bigvee X$.

Proposition 45. A complete distributive lattice $(A, \wedge, \vee, \bot, \top)$ is a Heyting algebra iff it satisfies the infinite distributivity law

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} a \wedge b_i.$$

More examples

- Every finite distributive lattice is a Heyting algebra.
- Every chain C with a least and greatest element is a Heyting algebra. For every $a, b \in C$ we have

$$a \to b = \begin{cases} \top & \text{if } a \leq b, \\ b & \text{if } a > b. \end{cases}$$

- Every Boolean algebra is a Heyting algebra.

Boolean algebras

For every element a of a Heyting algebra let $\neg a := a \to \bot$.

Proposition 49. Let $\mathfrak{A} = (A, \wedge, \vee, \to, \bot)$ be a Heyting algebra. Then the following three conditions are equivalent:

1. $\mathfrak{A}$ is a Boolean algebra;

2. $a \vee \neg a = \top$ for every $a \in A$;

3. $\neg\neg a = a$, for every $a \in A$.

The connection between Heyting algebras and Kripke frames

Let $\mathfrak{F} = (W, R)$ be an intuitionistic Kripke frame.

For every $w \in W$ and $U \subseteq W$ let

- $R(w) = \{v \in W : wRv\}$,
- $R^{-1}(w) = \{v \in W : vRw\}$,
- $R(U) = \bigcup_{w \in U} R(w)$,
- $R^{-1}(U) = \bigcup_{w \in U} R^{-1}(w)$.

Heyting algebras and Kripke frames, continued

A subset $U \subseteq W$ is called an upset if $w \in U$ and wRv implies $v \in U$.

Let $Up(\mathfrak{F})$ be the set of all upsets of $\mathfrak{F}$.

For $U, V \in Up(\mathfrak{F})$, let

$$U \to V = \{w \in W : \text{ for every } v \in W \text{ with } wRv \text{ if } v \in U \text{ then } v \in V\}$$

$$= W \setminus R^{-1}(U \setminus V).$$

Proposition. $(Up(\mathfrak{F}), \cap, \cup, \to, \emptyset)$ is a Heyting algebra.

General Frames

Let $\mathcal{A}$ be a set of upsets of $\mathfrak{F}$ closed under $\cap, \cup, \rightarrow$ and containing $\emptyset$.

$\mathcal{A}$ is a Heyting algebra.

A triple $\mathfrak{F} = (W, R, \mathcal{A})$ is called a general frame.

Descripitive Frames

- The duality does not generalize easily to general frames in general. We use the descriptive frames. They are general frames with two additional properties:
- $\mathfrak{F}$ is refined if $\forall w, v \in W, \neg(wRv) \Rightarrow \exists U \in \mathcal{A}(w \in U \wedge w \notin U)$,
- $\mathfrak{F}$ is compact if $\forall \mathcal{X} \subseteq \mathcal{A}, \forall \mathcal{Y} \subseteq \{W \backslash U \mid U \in \mathcal{A}\}(\mathcal{X} \cup \mathcal{Y}$ has the f.i.p. (finite intersection property)$\}$.
- Theorem. For every Heyting algebra $\mathfrak{A}$ there exists a descriptive frame $\mathfrak{A} = (W, R, mathcal A)$ such that $\mathfrak{A}$ is isomorphic to $(\mathcal{A}, \cup, \cap, \rightarrow, \emptyset)$.

The connection of Heyting algebras and topology

Definition 51. A pair $\mathcal{X} = (X, \mathcal{O})$ is called a topological space if $X \neq \emptyset$ and $\mathcal{O}$ is a set of subsets of X such that

- $X, \emptyset \in \mathcal{O}$
- If $U, V \in \mathcal{O}$, then $U \cap V \in \mathcal{O}$
- If $U_i \in \mathcal{O}$, for every $i \in I$, then $\bigcup_{i \in I} U_i \in \mathcal{O}$

For $Y \subseteq X$, the interior of Y is the set $\mathbf{I}(Y) = \bigcup\{U \in \mathcal{O} : U \subseteq Y\}$.

Heyting algebras and topology, continued

For every $U, V \in \mathcal{O}$ let

$$U \to V = \mathbf{I}((X \setminus U) \cup V)$$

Proposition. $(\mathcal{O}, \cup, \cap, \to, \emptyset)$ is a Heyting algebra.

Kripke frames from Heyting algebras

How to obtain a Kripke frame from a Heyting algebra?

Let $\mathfrak{A} = (A, \wedge, \vee, \to, \bot)$ be a Heyting algebra.

$F \subseteq A$ is called a filter if

- $a, b \in F$ implies $a \wedge b \in F$
- $a \in F$ and $a \leq b$ imply $b \in F$

A filter F is called prime if

- $a \vee b \in F$ implies $a \in F$ or $b \in F$

Kripke frames from Heyting algebras, continued

If $\mathfrak{A}$ is a Boolean algebra, then every prime filter of $\mathfrak{A}$ is maximal.

This is not the case for Heyting algebras.

Let $W := \{F : F \text{ is a prime filter of } \mathfrak{A}\}$.

For $F, F' \in W$ we say that FRF' if $F \subseteq F'$.

(W, R) is an intuitionistic Kripke frame.

Basic algebraic operations, homomorphisms

Let $\mathfrak{A} = (A, \wedge, \vee, \rightarrow, \bot)$ and $\mathfrak{A}' = (A', \wedge', \vee', \rightarrow', \bot')$ be Heyting algebras.

A map $h : A \rightarrow A'$ is called a Heyting homomorphism if

- $h(a \wedge b) = h(a) \wedge' h(b)$
- $h(a \vee b) = h(a) \vee' h(b)$
- $h(a \rightarrow b) = h(a) \rightarrow' h(b)$
- $h(\bot) = \bot'$

An algebra $\mathfrak{A}'$ is called a homomorphic image of $\mathfrak{A}$ if there exists a homomorphism from $\mathfrak{A}$ onto $\mathfrak{A}'$.

Basic algebraic operations, subalgebras

$\mathfrak{A}'$ is a subalgebra of $\mathfrak{A}$ if $A' \subseteq A$ and for every $a, b \in A'$ $a \wedge b, a \vee b, a \to b, \bot \in A'$.

A product $\mathfrak{A} \times \mathfrak{A}'$ of $\mathfrak{A}$ and $\mathfrak{A}'$ is the algebra $(A \times A', \wedge, \vee, \to, \bot)$, where

- $(a, a') \wedge (b, b') := (a \wedge b, a' \wedge' b')$
- $(a, a') \vee (b, b') := (a \vee b, a' \vee' b')$
- $(a, a') \to (b, b') := (a \to b, a' \to' b')$
- $\bot := (\bot, \bot')$

Categories

Let **Heyt** be a category whose objects are Heyting algebras and whose morphisms are Heyting homomorphisms.

Let **Kripke** denote the category of intuitionistic Kripke frames and p-morphisms.

We define $\varphi : \mathbf{Heyt} \to \mathbf{Kripke}$ and $\Psi : \mathbf{Kripke} \to \mathbf{Heyt}$.

$\mathfrak{A} \mapsto \varphi(\mathfrak{A}) = (W, R)$.

For a homomorphism $h : \mathfrak{A} \to \mathfrak{A}'$ let $\varphi(h) : \varphi(\mathfrak{A}') \to \varphi(\mathfrak{A})$ be such that for every element $F \in \varphi(\mathfrak{A}')$ we have $\varphi(h)(F) := h^{-1}(F)$.

Categories, continued

Define a functor $\Psi : \mathbf{Kripke} \to \mathbf{Heyt}$.

For every Kripke frame $\mathfrak{F}$ let $\Psi(\mathfrak{F}) = (Up(\mathfrak{F}), \cap, \cup, \rightarrow, \emptyset)$.

If $f : \mathfrak{F} \to \mathfrak{F}'$ is a p-morphism, then $\Psi(f) : \varphi(\mathfrak{F}') \to \varphi(\mathfrak{F})$ is such that for every element of $U \in \Psi(\mathfrak{F}')$ we have $\Psi(f)(U) = f^{-1}(U)$.

Duality

Theorem 57. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Heyting algebras and $\mathfrak{F}$ and $\mathfrak{G}$ Kripke frames.

1. - If $\mathfrak{A}$ is a homomorphic image of $\mathfrak{B}$, then $\varphi(\mathfrak{A})$ is isomorphic to a generated subframe of $\varphi(\mathfrak{B})$.
   - If $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, then $\varphi(\mathfrak{A})$ is a p-morphic image of $\varphi(\mathfrak{B})$.
   - If $\mathfrak{A} \times \mathfrak{B}$ is a product of $\mathfrak{A}$ and $\mathfrak{B}$, then $\varphi(\mathfrak{A} \times \mathfrak{B})$ is isomorphic to the disjoint union $\varphi(\mathfrak{A}) \uplus \varphi(\mathfrak{B})$.

Duality, continued

2. - If $\mathfrak{F}$ is a generated subframe of $\mathfrak{G}$, then $\Psi(\mathfrak{F})$ is isomorphic to a homomorphic image of $\Psi(\mathfrak{G})$.

   - If $\mathfrak{F}$ is a p-morphic image of $\mathfrak{G}$, then $\Psi(\mathfrak{F})$ is a subalgebra of $\Psi(\mathfrak{G})$.

   - If $\mathfrak{F} \uplus \mathfrak{G}$ is a disjoint union of $\mathfrak{F}$ and $\mathfrak{G}$, then $\Psi(\mathfrak{F} \uplus \mathfrak{G})$ is isomorphic to the product $\Psi(\mathfrak{F}) \times \Psi(\mathfrak{G})$.

Duality, continued 2

Is $\varphi(\mathbf{Heyt})$ isomorphic to **Kripke**?

Is $\Psi(\mathbf{Kripke})$ isomorphic to **Heyt**?

NO!

Duality, continued 3

$\Psi(\mathfrak{F}) = (Up(\mathfrak{F}), \cap, \cup, \rightarrow, \emptyset)$ is a complete lattice.

Not every Heyting algebra is complete.

Open question 62. Characterization of Kripke frames in $\Psi(\mathbf{Heyt})$.

Restrictions of φ and Ψ to the categories of finite Heyting algebras and finite Kripke frames respectively, are dually equivalent.

Theorem 63. For every finite Heyting algebra $\mathfrak{A}$ there exists a Kripke frame $\mathfrak{F}$ such that $\mathfrak{A}$ is isomorphic to $Up(\mathfrak{F})$.

Duality, continued, 4

For every Heyting algebra $\mathfrak{A}$ the algebra $\Psi\varphi(\mathfrak{A})$ is called a canonical extension of $\mathfrak{A}$. For every Kripke frame $\mathfrak{F}$ the frame $\varphi\Psi(\mathfrak{F})$ is called a prime filter extension of $\mathfrak{F}$. (Adaption needed for descriptive frames.)

Proposition.

- $\mathfrak{A}$ is a subalgebra of $\Psi\varphi(\mathfrak{A})$.
- $\mathfrak{F}$ is a p-morphic image of $\varphi\Psi(\mathfrak{F})$.
- $\mathfrak{A}$ is not isomorphic to a homomorphic image of $\Psi\varphi(\mathfrak{A})$.
- $\mathfrak{F}$ is not isomorphic to a generated subframe of $\varphi\Psi(\mathfrak{F})$.

Algebraic completeness

Let K be a class of algebras of the same signature.

We say that K is a variety if K is closed under homomorphic images, subalgebras and products.

Theorem. (Tarski) K is a variety iff $K = \mathbf{HSP}(K)$, where **H**, **S** and **P** are respectively the operations of taking homomorphic images, subalgebras and products.

Theorem 64. (Birkhoff) A class of algebras forms a variety iff it is equationally defined.

Heyt is a variety.

Valuations on Heyting algbras

Let $\mathcal{P}$ be the (finite or infinite) set of propositional variables.

Let $Form$ be the set of all formulas in this language.

Let $\mathfrak{A} = (A, \wedge, \vee, \rightarrow, \bot)$ be a Heyting algebra. A function $v : \mathcal{P} \rightarrow A$ is called a valuation into the Heyting algebra $\mathfrak{A}$.

We extend the valuation from $\mathcal{P}$ to the whole of $Form$ by putting:

- $v(\varphi \wedge \psi) = v(\varphi) \wedge v(\psi)$
- $v(\varphi \vee \psi) = v(\varphi) \vee v(\psi)$
- $v(\varphi \rightarrow \psi) = v(\varphi) \rightarrow v(\psi)$
- $v(\bot) = \bot$

Soundness

A formula φ is true in $\mathfrak{A}$ under v if $v(\varphi) = \top$.

φ is valid in $\mathfrak{A}$ if φ is true for every valuation in $\mathfrak{A}$.

Proposition 66.(Soundness) $\mathbf{IPC} \vdash \varphi$ implies that φ is valid in every Heyting algebra.

Completeness

Define an equivalence relation $\equiv$ on $Form$ by putting

$$\varphi \equiv \psi \quad \text{iff} \quad \vdash_{\mathbf{IPC}} \varphi \leftrightarrow \psi.$$

Let $[\varphi]$ denote the $\equiv$-equivalence class containing φ.

$Form/_{\equiv} := \{[\varphi] : \varphi \in Form\}$.

Define the operations on $Form/_{\equiv}$ by letting:

- $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$
- $[\varphi] \vee [\psi] = [\varphi \vee \psi]$
- $[\varphi] \to [\psi] = [\varphi \to \psi]$

Completeness 2

The operations on $Form/_{\equiv}$ are well-defined.

That is, if $\varphi' \equiv \varphi''$ and $\psi' \equiv \psi''$, then $\varphi' \circ \psi' \equiv \varphi'' \circ \psi''$, for $\circ \in \{\vee, \wedge, \to\}$.

Denote by $F(\omega)$ the algebra $(Form/_{\equiv}, \wedge, \vee, \to, \bot)$.

We call $F(\omega)$ the Lindenbaum-Tarski algebra of **IPC** or the ω-generated free Heyting algebra.

Completeness 3

Theorem 68.

1. $F(\alpha)$, for $\alpha \leq \omega$ is a Heyting algebra.

2. $\mathbf{IPC} \vdash \varphi$ iff φ is valid in $F(\omega)$.

3. $\mathbf{IPC} \vdash \varphi$ iff φ is valid in $F(n)$, for any formula φ in n variables.

Corollary 69. **IPC** is sound and complete with respect to algebraic semantics.

Jankov formulas and intermediate logics

Fix a propositional language $\mathcal{L}_n$ consisting of finitely many propositional letters $p_1, \ldots, p_n$ for $n \in \omega$.

Let $\mathfrak{M} = (W, R, V)$ be an intuitionistic Kripke model.

With every point w of $\mathfrak{M}$, we associate a sequence $i_1 \ldots i_n$ such that for $k = 1, \ldots, n$:

$$i_k = \begin{cases} 1 & \text{if } w \models p_k, \\ 0 & \text{if } w \not\models p_k \end{cases}$$

We call the sequence $i_1 \ldots i_n$ associated with w the color of w and denote it by $col(w)$.

Colors

Colors are ordered according to the relation $\leq$ such that $i_1 \ldots i_n \leq i'_1 \ldots i'_n$ if for every $k = 1, \ldots, n$ we have that $i_k \leq i'_k$.

The set of colors of length n ordered by $\leq$ forms an n-element Boolean algebra.

We write $i_1 \ldots i_n < i'_1 \ldots i'_n$ if $i_1 \ldots i_n \leq i'_1 \ldots i'_n$ and $i_1 \ldots i_n \neq i'_1 \ldots i'_n$.

Covers, anti-chains

For a Kripke frame $\mathfrak{F} = (W, R)$ and $w, v \in W$, we say that a point w is an immediate successor of a point v if $w \neq v$, vRw, and there is no $u \in W$ such that $u \neq v$, $u \neq w$, vRu and uRw.

We say that a set A totally covers a point v and write $v \prec A$ if A is the set of all immediate successors of v.

$A \subseteq W$ is an anti-chain if $|A| > 1$ and for every $w, v \in A$, if $w \neq v$ then $\neg(wRv)$ and $\neg(vRw)$

The construction of the n-universal model

The 2-universal model $\mathcal{U}(2) = (U(2), R, V)$ of **IPC** is the smallest Kripke model satisfying the following three conditions:

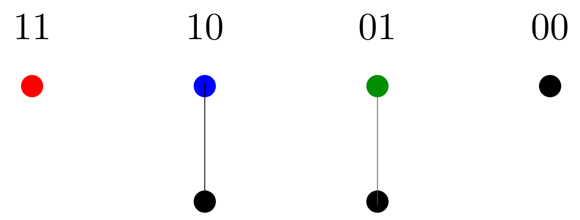
1. $max(\mathcal{U}(2))$ consists of 2^2 points of distinct colors.

2. If $w \in U(2)$, then for every color $i_1i_2 < col(w)$, there exists $v \in U(2)$ such that $v \prec w$ and $col(v) = i_1i_2$.

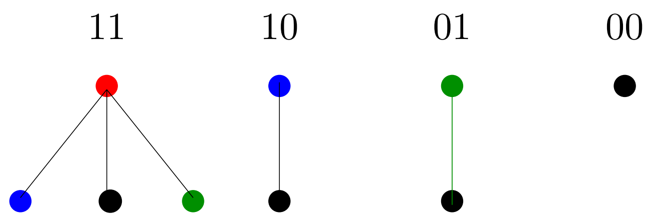
3. For every finite anti-chain $A \subset U(2)$ and every color i_1i_2, such that $i_1i_2 \leq col(u)$ for all $u \in A$, there exists $v \in U(2)$ such that $v \prec A$ and $col(v) = i_1i_2$.

The construction of the n-universal model

The construction of the n-universal model



The construction of the n-universal model



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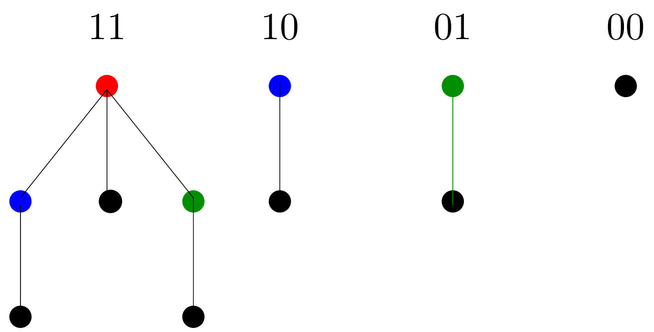

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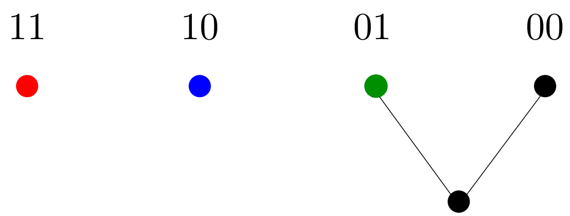

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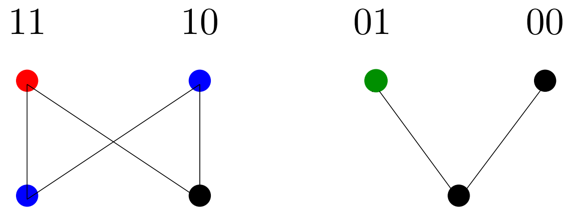

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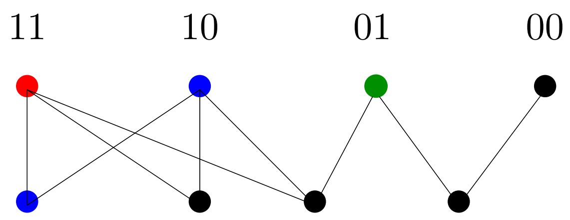

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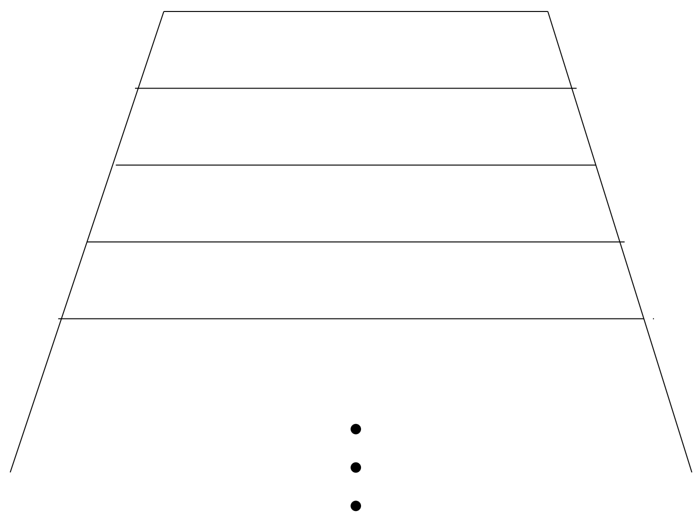
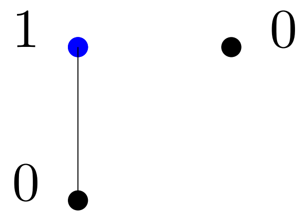
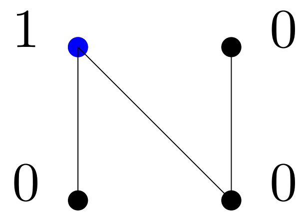


1-universal model

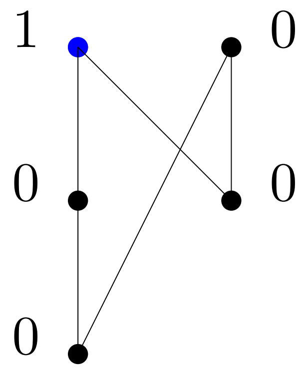
1-universal model



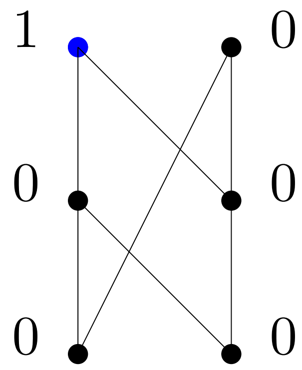
1-universal model



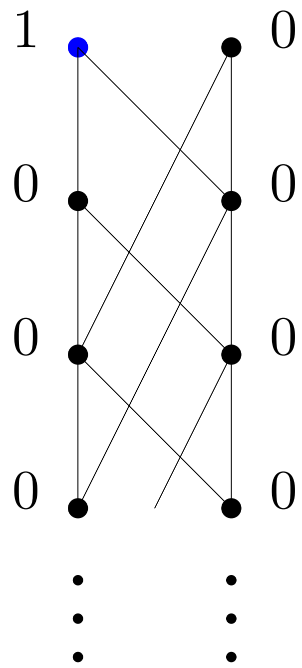
1-universal model



1-universal model



1-universal model



1-universal model is called the Rieger-Nishimura ladder.

Theorem. For every formula φ in the language $\mathcal{L}_n$, we have that

$$\vdash_{\mathbf{IPC}} \varphi \quad \text{iff} \quad \mathcal{U}(n) \models \varphi.$$

Call a set $V \subseteq U(n)$ definable or definable if there is a formula φ such that $V = \{w \in U(n) : w \models \varphi\}$.

Every upset of the Rieger-Nishimura ladder is definable.

Not every upset of the n-universal model (for $n > 1$) is definable.

Theorem 42. The Heyting algebra of all definable subsets of the n-universal model is isomorphic to the free n-generated Heyting algebra.

Point generated upsets of $\mathcal{U}(n)$

Which upsets of $\mathcal{U}(n)$ are definable?

All the point generated upsets of $\mathcal{U}(n)$ are definable.

For every formula $w \in U(n)$ we construct formulas φ_w and ψ_w such that φ_w defines $R(w)$ and ψ_w defines $U(n) \setminus R^{-1}(w)$.

Let w be a maximal point of $\mathcal{U}(n)$. Then

$$\varphi_w := \bigwedge\{p_k : w \models p_k\} \wedge \bigwedge\{\neg p_j : w \not\models p_j\} \text{ for each } k, j = 1, \ldots, n$$

and

$$\psi_w = \neg\varphi_w.$$

Point generated upsets of $\mathcal{U}(n)$, 2

If φ_w is defined, then

$$\psi_w := \varphi_w \rightarrow \bigvee_{i=1}^{n} \varphi_{w_i}$$

where $w_1, \ldots, w_n$ are all the immediate successors of w.

φ_w and ψ_w are called de Jongh formulas.

Point generated upsets of $\mathcal{U}(n)$, 3

- Let $prop(w) := \{p_k \mid w \models p_k\}$, the atoms true in w.
- $newprop(w) := \{p_k \mid w \not\models p_k \wedge \forall i \leqslant n(w_i \models p_k)\}$, the set of atoms which might have been true in w but aren't.
- $$\varphi(w) := \bigwedge prop(w) \wedge (\bigvee newprop(w) \vee \bigvee_{i=1}^{n} \psi_{w_i} \rightarrow \bigvee_{i=1}^{n} \varphi_{w_i})$$

Structure of n-universal model

Theorem 82.

- For every Kripke model $\mathfrak{M} = (\mathfrak{F}, V)$, there exists a Kripke model $\mathfrak{M}' = (\mathfrak{F}', V')$ such that $\mathfrak{M}'$ is a generated submodel of $\mathcal{U}(n)$ and $\mathfrak{M}'$ is a p-morphic image of $\mathfrak{M}$.
- For every finite Kripke frame $\mathfrak{F}$, there exists a valuation V, and $n \leq |\mathfrak{F}|$ such that $\mathfrak{M} = (\mathfrak{F}, V)$ is a generated submodel of $\mathcal{U}(n)$.

n-Henkin model and n-universal model

Theorem The n-universal model is isomorphic to the nodes of finite depth in the n-canonical model.

n-Henkin model and n-universal model

- Proof By induction on the depth of the nodes it is shown that the submodel generated by a node in the n-Henkin model is isomorphic to the submodel generated by some node in the n-Henkin model and vice versa.
- $\Longrightarrow$: The p-morphism guaranteed by Theorem 82 from the finite Henkin model into the universal model has to be an isomorphism since all nodes of a Henkin model have distinct theories.
- $\Longleftarrow$: For nodes of depth 1 this is trivial. Consider a node of depth $n+1$. From the induction hypothesis one sees that there is a p-morphism from the nodes of depth $\leqslant n$. For nodes of depth $n+1$ one has to use the de Jongh formula to see that in the n-Henkin model can be only one node above the image of depth $n+1$ which satisfies the formula.

The Jankov theorem

Theorem 87. For every finite rooted frame $\mathfrak{F}$ there exists a formula $\chi(\mathfrak{F})$ such that for every frame $\mathfrak{G}$

$$\mathfrak{G} \not\models \chi(\mathfrak{F}) \text{ iff } \mathfrak{F} \text{ is a } p\text{-morphic image of a generated subframe of } \mathfrak{G}.$$

Lemma 88. A frame $\mathfrak{F}$ is a p-morphic image of a generated subframe of a frame $\mathfrak{G}$ iff $\mathfrak{F}$ is a generated subframe of a p-morphic image of $\mathfrak{G}$.

Congruence extension property

Proof. It is a universal algebraic result that if a variety **V** has the congruence extension property, then for every algebra $\mathfrak{A} \in \mathbf{V}$ we have that $\mathbf{HS}(\mathfrak{A}) = \mathbf{SH}(\mathfrak{A})$.

Heyt has the congruence extension property.

The result follows from the duality of Heyting algebras and Kripke frames.

Theorem 87(Reformulated). For every finite rooted frame $\mathfrak{F}$ there exists a formula $\chi(\mathfrak{F})$ such that for every frame $\mathfrak{G}$ $\mathfrak{G} \not\models \chi(\mathfrak{F})$ iff $\mathfrak{F}$ is a generated subframe of a p-morphic image of $\mathfrak{G}$.

Proof of the Jankov theorem

Let $\mathfrak{F}$ be a finite rooted frame.

Then exists $n \in \omega$ such that $\mathfrak{F}$ is (isomorphic to) a generated subframe of $\mathcal{U}(n)$.

Let $w \in U(n)$ be the root of $\mathfrak{F}$. Then $\mathfrak{F}$ is isomorphic to $\mathfrak{F}_w$.

Let $\chi(\mathfrak{F}) := \psi_w$.

Proof of the Jankov theorem, 2

$\mathfrak{F} \not\models \psi_w$ hence if $\mathfrak{F}$ is a generated subframe of a p-morphic image of $\mathfrak{G}$ then $\mathfrak{G} \not\models \psi_w$.

If $\mathfrak{G} \not\models \psi_w$, then there is a valuation V such that model $\mathfrak{M} \not\models \psi_w$, where $\mathfrak{M} = (\mathfrak{G}, V)$.

We can assume that there is a p-morphic image $\mathfrak{M}'$ of of $\mathfrak{M}$ such that $\mathfrak{M}'$ is a generated submodel of $\mathcal{U}(n)$.

Then $\mathfrak{M}' \not\models \psi_w$. Which implies that $\mathfrak{F}_w$ is a generated subframe of $\mathfrak{M}'$.

Applications of Jankov formulas

Let $\mathfrak{F}$ and $\mathfrak{G}$ be Kripke frames.

We say that

$\mathfrak{F} \leq \mathfrak{G}$ if $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$.

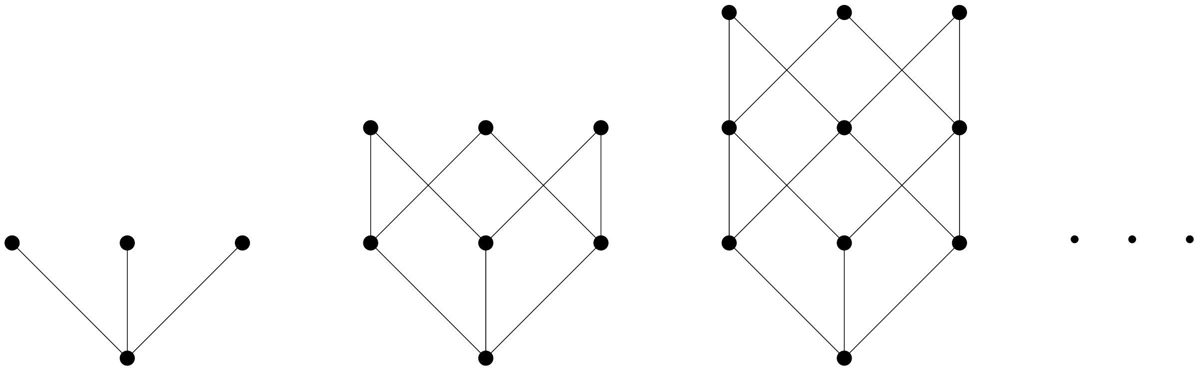
- $\leq$ is reflexive and transitive.
- If we restrict ourselves to only finite Kripke frames, then $\leq$ is a partial order.
- In the infinite case $\leq$, in general, is not antisymmetric.

The Jankov chain

Let $\mathfrak{F}$ and $\mathfrak{F}'$ be two finite rooted frames.

If $\mathfrak{F} \leq \mathfrak{F}'$, then for every frame $\mathfrak{G}$ we have that $\mathfrak{G} \models \chi(\mathfrak{F})$ implies $\mathfrak{G} \models \chi(\mathfrak{F}')$.

Consider the sequence Δ of finite Kripke frames shown below



Properties of the Jankov chain

Lemma 91. Δ forms a $\leq$-antichain.

For every set Γ of Kripke frames. Let $Log(\Gamma)$ be the logic of Γ, that is, $Log(\Gamma) = \{\varphi : \mathfrak{F} \models \varphi$ for every $\mathfrak{F} \in \Gamma\}$.

Theorem 92. For every $\Gamma_1, \Gamma_2 \subseteq \Delta$, if $\Gamma_1 \neq \Gamma_2$, then $Log(\Gamma_1) \neq Log(\Gamma_2)$.

Proof. Without loss of generality assume that $\Gamma_1 \not\subseteq \Gamma_2$.

This means that there is $\mathfrak{F} \in \Gamma_1$ such that $\mathfrak{F} \notin \Gamma_2$.

Consider the Jankov formula $\chi(\mathfrak{F})$.

Then $\mathfrak{F} \not\models \chi(\mathfrak{F})$.

Properties of the Jankov chain, 2

Therefore, $\Gamma_1 \not\models \chi(\mathfrak{F})$ and $\chi(\mathfrak{F}) \notin Log(\Gamma_1)$.

Now we show that $\chi(\mathfrak{F}) \in Log(\Gamma_2)$.

Suppose $\chi(\mathfrak{F}) \notin Log(\Gamma_2)$.

Then there is $\mathfrak{G} \in \Gamma_2$ such that $\mathfrak{G} \not\models \chi(\mathfrak{F})$.

This means that $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$.

Hence, $\mathfrak{F} \leq \mathfrak{G}$ which contradicts the fact that Δ forms a $\leq$-antichain.

Therefore, $\chi(\mathfrak{F}) \notin Log(\Gamma_1)$ and $\chi(\mathfrak{F}) \in Log(\Gamma_2)$.

Thus, $Log(\Gamma_1) \neq Log(\Gamma_2)$.

Continuum many logics

Corollary 93. There are continuum many intermediate logics.