# Diego's Theorem

Dick de Jongh/ ILLC, Universiteit van Amsterdam

February 28, 2017

**Theorem 1** (Diego,1961(!))**.** The fragment $[\to]$ of formulas of $\mathsf{IPC}$ with only $\to$ is locally finite, i.e. the number of equivalence classes of formulas in a finite number of variables is finite.

The proof takes a number of steps. The proof is sketched for formulas in $[\to, \wedge, \bot]$, which is not more difficult. An $n$-model is a Kripke model for the propositional vriables $p_0, \ldots, p_{n-1}$.

**Definition 2.** A world $w$ in a Kripke model $\mathfrak{M}$ is an *intersection node* if $w$ is not terminal, and for each proposition letter $p$, $w \in V(p)$ iff $w' \in V(p)$ for each $w'$ with $wR^+w'$.
We write $uR^+v$ for $uRv$ and $u \neq v$.

This means that, if $p$ fails at an intersection node it fails higher up in the model.

**Proposition 3.** If $w$ is an intersection node, then, for all formulas $\varphi$ in $[\to, \wedge, \bot]$, $w \Vdash \varphi$ iff $w' \Vdash \varphi$ for each $w'$ with $wR^+w'$.

*Proof.* By induction on the length of $\varphi$. We just do the cases of $\wedge$ and $\to$.
$w \Vdash \psi \wedge \chi$ iff $w \Vdash \psi$ and $w \Vdash \chi$ iff (by IH) $w' \Vdash \psi$ for each $w'$ with $wR^+w'$ and $w' \Vdash \chi$ for each $w'$ with $wR^+w'$ iff $w' \Vdash \psi \wedge \chi$ for each $w'$ with $wR^+w'$.
$w \nVdash \psi \to \chi$ iff ($w \vDash \psi$ and $w \nvDash \chi$) or $\exists w'(wR^+w'$ and $w' \vDash \psi$ and $w' \nvDash \chi)$ iff (by IH) ($w \vDash \psi$ and $\exists w'(wR^+w'$ and $w' \nvDash \chi)$ or $\exists w'(wR^+w'$ and $w' \vDash \psi$ and $w' \nvDash \chi)$. In both cases this implies $\exists w'(wR^+w'$ and $w' \nvDash \psi \to \chi)$, and the other direction is trivial. □

**Proposition 4.** If $\mathfrak{M}$ is a finite rooted Kripke model and $\mathfrak{M}'$ arises from $\mathfrak{M}$ by deleting all intersection nodes but the root, then, for all formulas $\varphi$ in $[\to, \wedge, \bot]$, and all $w$ in $\mathfrak{M}'$, $\mathfrak{M}', w \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \varphi$.

*Proof.* Induction on the length of $\varphi$. The case of $\varphi = \psi \to \chi$ runs as follows: Take $w$ in $\mathfrak{M}'$. $\mathfrak{M}, w \nVdash \varphi$ iff, (for some non-intersection node $w'$ with $wRw'$, $\mathfrak{M}, w' \Vdash \psi$ and $\mathfrak{M}, w' \nVdash \chi$) or (for some intersection node $w'$ with $wRw'$, $\mathfrak{M}, w' \Vdash \psi$ and $\mathfrak{M}, w' \nVdash \chi$). Now, the latter is, by repeatedly applying Proposition **??** noting that the model is finite and intersection nodes on non-terminal, to: for some intersection node $w'$ with $wRw'$, $\mathfrak{M}, w' \Vdash \psi$ and $\mathfrak{M}, w' \nVdash \chi$ iff, by induction hypothesis, $\mathfrak{M}', w \nVdash \varphi$. □

Note that an $n$-model without intersection nodes the number of atoms forced in a chain running from top to bottom decreases at each step, so that chains cannot be longer than $n$, i.e. the depth of an $n$-model without intersection nodes is bounded by $n$. We have now proved that, if two $n$-formulas $\varphi, \psi \in [\to, \wedge, \bot]$ are not $\mathsf{IPC}$-equivalent and hence can be distinguished on a finite model, then they can be distinguished on a model of depth $n$ or smaller.
We now sketch a technique which will be made more precise when we discuss universal models to obtain the fact there is a finite set of models that distinguishes all formulas in $[\to, \wedge, \bot]$, and hence this class of formulas contains only a finite number of equivalence classes. We need think only of the case that a model distinguishes two formulas in its root. Also, we need consider only models that are trees.
We call a function from $(p_0, \ldots, p_{n-1})$ to $\{0, 1\}$ a *color*. There are $2^n$ such colors, so only $2^n$ non-isomorphic rooted $n$-models of depth 1. Suppose we have a class $X$ of $n$-models of depth $m$

that suffices to distinguish all pairs of formulas that can be distinguished on a model of depth $m$. Then a class of models of depth $m+1$ can be obtained with the same property by putting a root with a fitting color below a subset of $X$.