Intuitionistic Logic

Exercise sheet 8

December 12, 2017

Exercise 1. [Heyting algebras]
We have defined HA's as posets with extremal elements, and in which every two elements a, b have a meet $a \wedge b$ (greatest lower bound), a join $a \vee b$ (least upper bound), and an implication $a \to b$ (an element s.t. $a \wedge c \leq b \iff c \leq a \to b$).

Sometimes (e.g., in the lecture notes on the course webpage) HA's are required to satisfy in addition the following distributive laws:

- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Show that the validity of these laws actually follows from the HA properties.
Hint: use the properties of implication to prove the first law; then use the first law to prove the second.

Exercise 2. [Lindenmaum-Tarski algebra]
Consider the Lindenbaum-Tarski algebra of IPC. We proved in class that for every formulas φ, ψ, we have $\overline{\varphi} \wedge \overline{\psi} = \overline{\varphi \wedge \psi}$, that is, the meet of the classes $\overline{\varphi}$ and $\overline{\psi}$ exists and coincides with the class $\overline{\varphi \wedge \psi}$. Prove the analogous characterizations for join and implication:

- $\overline{\varphi} \vee \overline{\psi} = \overline{\varphi \vee \psi}$
- $\overline{\varphi} \to \overline{\psi} = \overline{\varphi \to \psi}$

Exercise 3. [Frame definability and intermediate logics]
Recall that the Kreisel-Putnam axiom KP is the following formula:

$$(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$$

The Kreisel-Putnam logic, $\mathbf{KP}$, is the intermediate logic axiomatized by KP. Mevedev's logic, $\mathbf{ML}$, is defined as the logic of all frames $\mathfrak{F}$ which are finite Boolean algebras without the top element, i.e. $\langle \wp(X) - \{\emptyset\}, \supseteq \rangle$ for a finite X.

Show that for any finite frame $\mathfrak{F}$, $\mathfrak{F} \Vdash \mathsf{KP} \iff \mathfrak{F}$ satisfies I-saturation:

(I-sat) $\quad \forall x \forall y, y' \in R[x] \exists z \in R[x]$ s.t. zRy and zRy' and $E_z = E_y \cup E_{y'}$

where E_w denotes the set of endpoints accessible from w. Use this characterization to show that $\mathbf{KP} \subseteq \mathbf{ML}$.