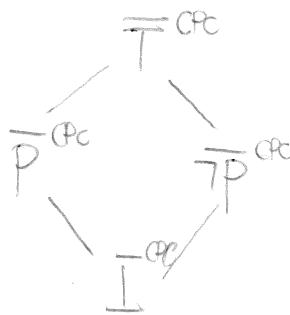


Class 9: Rieger-Nishimura lattice & ladder

$$L_p/\text{CPC} = \left\{ \bar{\varphi}^{\text{CPC}} \mid \varphi \text{ is a formula in } \begin{array}{l} \text{the prop. variable } p \end{array} \right\}$$

$$L_p/\text{CPC} = \{ \bar{T}^{\text{CPC}}, \bar{I}^{\text{CPC}}, \bar{P}^{\text{CPC}}, \bar{\neg P}^{\text{CPC}} \}$$

The Lindenbaum-Tarski algebra LT_{CPC}^p of CPC in one prop. letter p is:



Theor. $L_{p_1, \dots, p_n}/\text{CPC}$ has 2^{2^n} elements.

Proof An eq. class $\bar{\varphi}^{\text{CPC}}$ is determined by a set of valuations — those that make φ true.

Conversely, each set of valuations determines an equivalence class. Thus, the elements of $L_{p_1, \dots, p_n}/\text{CPC}$ are in 1-1 correspondence with the sets of valuations for p_1, \dots, p_n . There are 2^n valuations for p_1, \dots, p_n , and so 2^{2^n} such sets. \square

Def A prop. logic L is locally finite if the $\mathbb{N}, L_{p_1, \dots, p_n}/L$ is finite.

Intuitively, L is locally finite iff with finitely many atoms we can express only finitely many meanings.

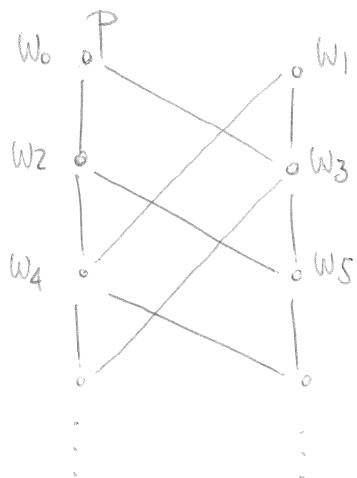
Example CPC is locally finite.

Our main aim in this class is to show:

Theor. IPC is not locally finite.

That is, in IPC even with finitely many atoms, in fact even a single atom, we can express infinitely many non-equivalent meanings. For this we consider a frame:

the Rieger-Nishimura ladder



$$\varphi_0 = \vartheta$$

$$\varphi_1 = 75^\circ$$

$$\varphi_2 = \pi p$$

$$\varphi_3 = \pi p \rightarrow p$$

$$\varphi_{n+4} = \varphi_{n+3} \rightarrow \varphi_n \vee \varphi_{n+1}$$

Prop For all $n \in \mathbb{N}$: $RN, w_i \vdash q_n \Leftrightarrow w_n R w_i$

Proof Induction on n.

- $n=0, 1, 2, 3$: check directly.
 - inductive step.

Suppose claim holds for $k < h+q$.

We show it holds for $k = n+q$ as well.

Consider the case n odd (the case for even is similar)

The configuration looks like this:

Suppose $v \notin R[W_{n+q}]$

Then vRw_{n+3} .

By IH, $W_{n+3} \vdash \varphi_{n+3}$ but $\nVdash \varphi_n \vee \varphi_{n+1}$.

Therefore $V \Vdash \varphi_{n+3} \rightarrow \varphi_n \vee \varphi_{n+1}$, i.e., $V \Vdash \varphi_n$.

Conversely, suppose $V \Vdash \varphi_{n+q}$.

Then vRu for some $u \in \varphi_{n+3}$, $u \notin \varphi_n \vee \varphi_{n+1}$.

The only such a is w_{n+3} .

But if $V \notin R[W_{n+3}]$ it follows that $V \notin R[W_{n+4}]$

Cor For $n \neq m$, $\varphi_n \not\equiv_{IPC} \varphi_m$

Notice that for all $n \in \mathbb{N}$, $\varphi_n \in L_p$, since only the letter p occurs in φ_n .

This means that L_p/IPC is infinite.

Cor IPC is not locally finite.

The next natural question is:

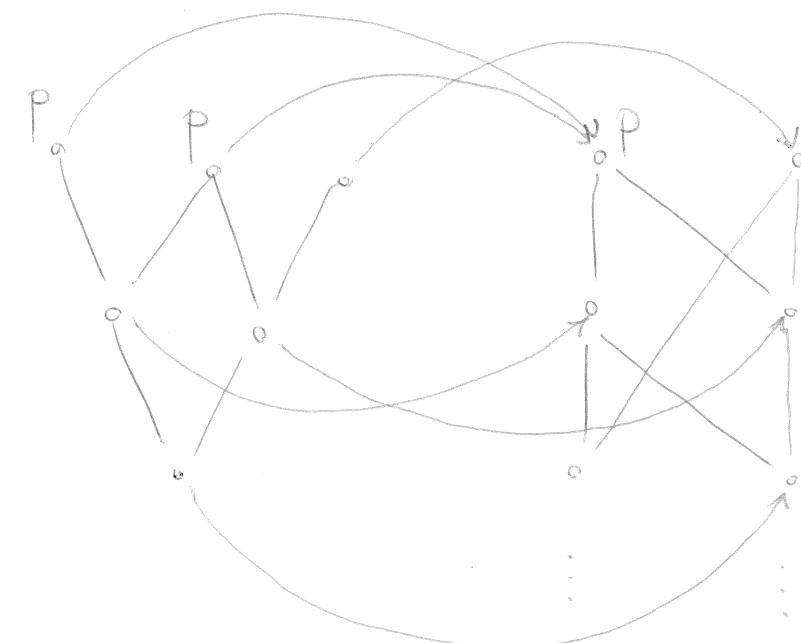
what does the algebra LT_{IPC}^p
(the Lindenbaum-Tarski algebra
of the one-letter fragment of IPC)
look like? To answer this question

we can once more use the RN
ladder, showing that it is "universal"
in the sense that it makes all pos-
sible distinctions between formulas.

Theor (Universality of the RN ladder)

For every finite rooted iKM M for L_p^f
there is a unique p -morphism $f: M \rightarrow RN$.

Proof By induction on the depth of M ,
ie., the maximum length of an antichain in M .
I will just look at an example. Proceed indu-
ctively from the endpoints to the root.



Cor Let φ be a formula in the prop. letter p . If $\varphi \notin \text{IPC}$ then φ can be falsified in RN.

Let us denote by $|\varphi|_{RN}$ the set of points in RN where φ is forced.

Prop For $\varphi, \psi \in L_p$: $\varphi \equiv_{\text{IPC}} \psi \Leftrightarrow |\varphi|_{RN} = |\psi|_{RN}$

Proof

\Rightarrow Obvious

\Leftarrow If $\varphi \not\equiv_{\text{IPC}} \psi$ then either $\varphi \rightarrow \psi \notin \text{IPC}$, or $\psi \rightarrow \varphi \notin \text{IPC}$. W.l.o.g. suppose the former. Then $\varphi \rightarrow \psi$ can be falsified in RN, so there is a point w in RN s.t. $w \Vdash \varphi$ and $w \not\Vdash \psi$. Thus, $|\varphi|_{RN} \neq |\psi|_{RN}$. \square

Cor L_{IPC}^T is isomorphic to $\langle U_p(\text{RN}), \subseteq \rangle$ via the function $\bar{\varphi}^{\text{IPC}} \mapsto |\varphi|_{RN}$.

Proof

- f is order-preserving:

$$\bar{\varphi}^{\text{IPC}} \leq \bar{\psi}^{\text{IPC}} \Leftrightarrow \varphi \vdash_{\text{IPC}} \psi \Leftrightarrow |\varphi|_{\text{IPC}} \subseteq |\psi|_{\text{IPC}}$$

- f is injective:

by the previous lemma $\bar{\varphi}^{\text{IPC}} \neq \bar{\psi}^{\text{IPC}} \Rightarrow |\varphi|_{RN} \neq |\psi|_{RN}$

- f is surjective:

if $U \in U_p(\text{RN})$ then one of the following holds:

$$1. U = \emptyset \quad \Rightarrow U = |\varnothing|_{RN}$$

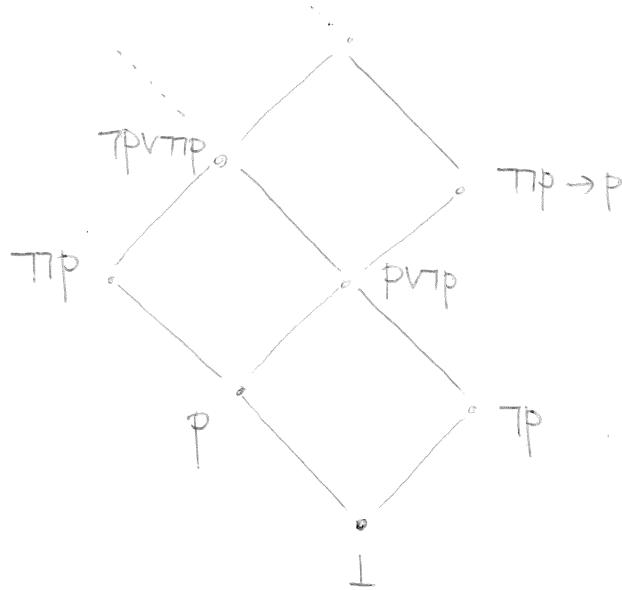
$$2. U = W \quad \Rightarrow U = |T|_{RN}$$

$$3. U = R[w_i] \quad \Rightarrow U = |\varphi_i|_{RN}$$

$$4. U = R[w_i] \cup R[w_{i+1}] \quad \Rightarrow U = |\varphi_i \vee \varphi_{i+1}|_{RN}$$

in each case $U = |\varphi|_{RN}$ for some φ . \square

By looking at the up-sets of RN we can see that the algebra LT_{IPC}^P looks as follows:



From the picture we immediately obtain some interesting info about the one-letter fragment of IPC, for instance:

Cor There is no consistent formula in L_p which properly entails p . (same for $\neg p$).

Cor For any $\varphi \in L_p$, $\varphi \neq T$, there are only finitely many non-equivalent formulas in L_p which entail φ .