

Class 10 : IQC

So far we looked at propositional logic.
Now let's turn to int. predicate logic, IQC.

BHK interpretation of quantifiers

- A proof of $\exists x\varphi(x)$ is an object d together with a proof of $\varphi(d)$
- A proof of $\forall x\varphi(x)$ is a method which, applied to an object d in the intended domain, returns a proof of $\varphi(d)$.

Rem

- A proof of $\forall x\exists y\varphi(x,y)$ yields a Skolem function f s.t. $\forall x\varphi(x,f(x))$, f computable

Def (Language of IQC)

Let P be a set of predicate symbols (each with a corresponding arity $n \in \mathbb{N}$). The language \mathcal{L}_P^Q is:

$$\begin{aligned}\varphi ::= & P(x_1, \dots, x_n) \mid x = y \mid \varphi \rightarrow \varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \perp \mid \\ & \forall x \varphi \mid \exists x \varphi\end{aligned}$$

where P is an n -ary predicate, x_1, \dots, x_n, x, y variables.
As in IPC, we define:

$$\neg \varphi := \varphi \rightarrow \perp$$

Natural deduction for IQC

We have the IPC rules for connectives plus the following rules.

I assume that y is a variable free for x in the formula φ .

intro

elim.

\exists

$$\frac{\varphi[y/x]}{\exists x. \varphi}$$

$$\frac{\exists x. \varphi \quad \frac{\psi}{\varphi}}{\psi} \quad \text{provided } y \text{ does not occur free in } \psi \text{ or any undisch. assumption besides } \varphi[y/x]$$

\forall

$$\frac{\varphi[y/x] \quad \text{provided } y \text{ does not occur free in any undischarged assumption}}{\forall x \varphi}$$

$$\frac{y=y' \quad \varphi[y/x]}{\varphi[y'/x]}$$

=

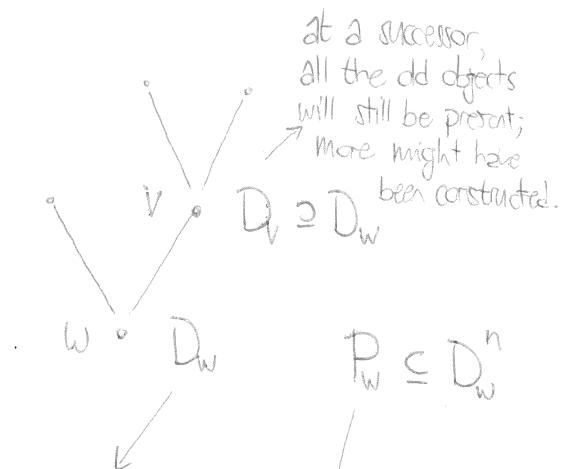
Def Write $\bar{\Phi} \vdash_{IQC} \psi$ if ψ is derivable from assumptions in $\bar{\Phi}$ in this system.

$$\underline{\text{Ex. } \forall x \neg P_x \equiv_{IQC} \neg \exists x P_x}$$

$$\frac{\forall x \neg P_x}{\neg P_y} \frac{\neg P_y}{\perp} \frac{\perp}{\perp} \frac{\perp}{\neg \exists x P_x} \frac{\neg \exists x P_x}{\perp} \frac{\perp}{\neg \exists x P_x} \quad \text{Je}$$

$$\frac{[Py]}{\exists x P_x} \frac{\exists x P_x}{\neg \exists x P_x} \frac{\neg \exists x P_x}{\perp} \frac{\perp}{\neg Py} \frac{\neg Py}{\forall x \neg P_x} \quad \text{Ti}$$

Kripke semantics for IQC



domain at w:
set of objects
that have been
constructed at w

extension of P
at w; set of
n-tuples which
have been proved
to stand in the
relation P at w

all tuples which had
already been proved to
stand in a relation still are,
→ and perhaps more.
 $P_v \supseteq P_w$ $\sim_v \supseteq \sim_w$

$P_w \subseteq D_w^n$

$\sim_w \subseteq D_w^2$

identity at w:
set of pairs of ob-
jects which have
been proved identi-
cal at w. Notice that
if $d \neq d'$ this doesn't
mean that d and d'
are distinct: they may
later be proved to be
identical.

Def An IQC-model is a tuple:

$$M = \langle W, R, \{D_w \mid w \in W\}, \{P_w \mid w \in W, P \in \Phi\}, \{\sim_w \mid w \in W\} \rangle$$

Where:

- $\langle W, R \rangle$ is a partial order
- $P_w \subseteq D_w^n$ if P is an n-ary pred. symbol
- $D_w \neq \emptyset$
- $\sim_w \subseteq D_w \times D_w$ is a congruence
i.e., a relation of equivalence (reflexive, symmetric & transitive)
such that $d_1 \sim_w d'_1, \dots, d_n \sim_w d'_n \Rightarrow [d_1, \dots, d_n] \in P_w \text{ iff } [d'_1, \dots, d'_n] \in P_w$

if wRw' we require:

- $D_w \subseteq D_{w'}$
- $P_w \subseteq P_{w'}$
- $\sim_w \subseteq \sim_{w'}$

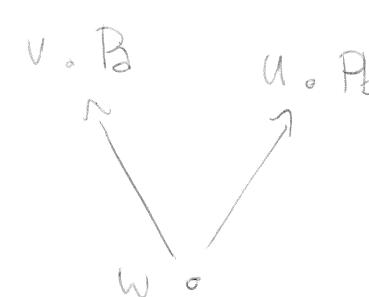
Def An assignment into D_w is a map $g: \text{Ibr} \rightarrow D_w$.

Example Show that $\neg \forall x P_x \not\vdash_{\text{IQC}} \exists x \neg P_x$

Rem If $w R w'$ an assignment into D_w is also into $D_{w'} \supseteq D_w$. So assignments can be "inherited" by w' .

Def (int. Kripke semantics for IQC)

- $M, w \Vdash_g P(x_1, \dots, x_n) \Leftrightarrow \langle g(x_1), \dots, g(x_n) \rangle \in P_w$
- $M, w \Vdash_g x = y \Leftrightarrow g(x) \sim_w g(y)$
- connectives as in IPC
- $M, w \Vdash_g \exists x \varphi \Leftrightarrow \text{for some } d \in D_w: M, w \Vdash_{g[x \mapsto d]} \varphi$
- $M, w \Vdash_g \forall x \varphi \Leftrightarrow \text{for all } w' \in R(w), \text{ for all } d \in D_{w'},$
 $M, w' \Vdash_{g[x \mapsto d]} \varphi$



Domains: $D_w = D_v = D_u = \{a, b\}$

Predicate: $P_w = \emptyset$

$P_v = \{a\}$

$P_u = \{b\}$

$$\left. \begin{array}{l} w \not\models_{[x \mapsto a]} \neg P_x \\ w \not\models_{[x \mapsto b]} \neg P_x \end{array} \right\} \Rightarrow w \not\models \exists x \neg P_x$$

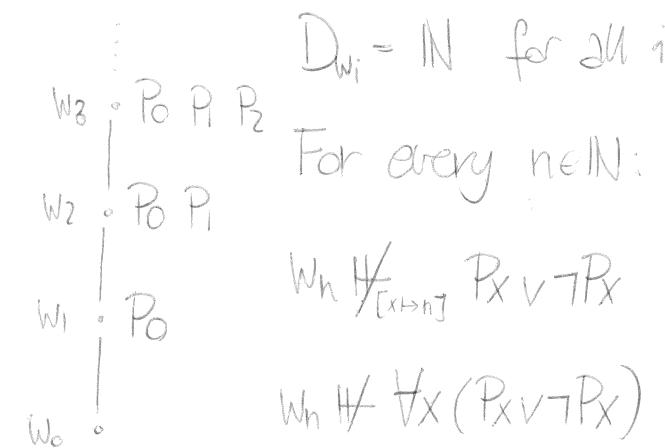
$$\left. \begin{array}{l} \text{there is no } w' \\ \text{s.t. } w' \models \forall x P_x \end{array} \right\} \Rightarrow w \models \neg \forall x P_x$$

Prop (basic features of the semantics)

- Persistency: $M, w \Vdash_g \varphi \wedge w R w' \Rightarrow M, w' \Vdash_g \varphi$
- Endpoints are complete:
if e is an endpoint then $M, e \Vdash_g \varphi$ or $M, e \Vdash_g \neg \varphi$
- Endpoints behave classically:
if $\varphi \in \text{CQC}$ and e is an endpoint, $M, e \Vdash_g \varphi$
- Invariance under generated submodels:
 $M, w \Vdash_g \varphi \Leftrightarrow M_w, w \Vdash_g \varphi$

$$\begin{aligned}
 (*) \quad \varphi \equiv_{\text{CPC}} \perp &\Leftrightarrow \neg \varphi \in \text{CPC} \\
 &\xrightarrow{\text{Gillivano}} \neg \neg \varphi \in \text{IPC} \\
 &\xrightarrow{\text{IPC}} \neg \varphi \in \text{IPC} \Leftrightarrow \varphi \equiv_{\text{IPC}} \perp
 \end{aligned}$$

Ex. An important example.



So, $w_0 \not\models \underbrace{\neg \forall x (Px \vee \neg Px)}$

Classical
Contradiction!

Rem φ contrad in CQC $\not\Rightarrow \varphi$ contrad. in IPC

This contrasts with the situation in prop. logic:

Prop φ contrad. in CPC $\Leftrightarrow \varphi$ contradiction in IPC

Proof (*)

Rem Glivenko's thm fails in IQC:

$$\varphi \in CPC \not\Rightarrow \pi\varphi \in IQC$$

Counterexample

Take the model in the previous example.

We have $w_0 \Vdash \neg \forall x(Px \vee \neg Px)$.

Thus, $w_0 \not\Vdash \forall x(Px \vee \neg Px)$.

This shows that we have:

$$\forall x(Px \vee \neg Px) \in CPC$$

$$\neg \forall x(Px \vee \neg Px) \notin IPC$$

The reason underlying the failure of Glivenko's thm is that π does not commute with \forall (contrast with $\pi(\varphi \wedge \psi) \equiv \pi\varphi \wedge \pi\psi$).

Prop. $\forall x \neg \forall x \Vdash_{IQC} \neg \forall x \forall x$

Proof Take again the model of the previous ex.:

w_3	$\vdash P_0 P_1 P_2$	$\forall i, n : w_i \not\Vdash_{[x \mapsto n]} \neg \forall x$
w_2	$\vdash P_0 P_1$	$\forall i, n : w_i \not\Vdash_{[x \mapsto n]} \neg \forall x$
w_1	$\vdash P_0$	$\forall i, n : w_i \not\Vdash_{[x \mapsto n]} \neg \forall x$
w_0		$\text{Thus: } w_0 \Vdash \forall x \neg \forall x$

$$\forall i : w_i \not\Vdash \forall x \forall x$$

$$\text{Thus: } w_0 \Vdash \neg \forall x \forall x$$

$$w_0 \not\Vdash \neg \forall x \forall x$$

Prop $\vdash x \pi \varphi \rightarrow \pi \vdash x \varphi$ is valid on finite models.

Proof. Let M be finite (i.e., W is finite).

Suppose $M, W \not\vdash \pi \vdash x \varphi$.

Then for some endpoint $v \in R[W]$, $M, V \vdash \pi \vdash x \varphi$.

So $M, V \not\vdash \vdash x \varphi$, which means that $M, V \not\vdash_{[x \mapsto d]} \varphi$

for some $d \in D$. By the completeness of endpoints,

$M, V \vdash_{[x \mapsto d]} \neg \varphi$, so $M, V \vdash_{[x \mapsto d]} \pi \neg \varphi$.

Thus, $M, W \not\vdash \vdash x \pi \varphi$. \square

Cor IQC does not have the finite model prop.

Proof $\vdash x \pi P_x \rightarrow \pi \vdash x P_x$ is not valid in IQC, yet it has no finite countermodel.

Def (Negative translation of CQC into IQC)

- $\varphi^n = \pi \varphi$ if φ is atomic
- $\perp^n = \perp$
- $(\varphi \wedge \psi)^n = \varphi^n \wedge \psi^n$
- $(\varphi \rightarrow \psi)^n = \varphi^n \rightarrow \psi^n$
- $(\varphi \vee \psi)^n = \neg(\neg \varphi^n \wedge \neg \psi^n)$
- $(\vdash x \varphi)^n = \vdash x \varphi^n$
- $(\exists x \varphi)^n = \neg \vdash x \neg \varphi^n$

Theorem $\varphi_1, \dots, \varphi_k \vdash_{\text{CQC}} \psi \Leftrightarrow \varphi_1^n, \dots, \varphi_k^n \vdash_{\text{IQC}} \psi^n$

Proof idea

- Show that $\vdash X: \chi^n \equiv_{\text{IQC}} \pi \chi^n$ (induction on χ)
- Using the previous point, show that a proof P of $\varphi_1, \dots, \varphi_k \vdash \psi$ in CQC can be translated to a proof P^n of $\varphi_1^n, \dots, \varphi_k^n \vdash \psi^n$ in IQC. Induct. on P . \rightarrow

Key step: simulate the Π rule.

$$\frac{\varphi_1 \dots \varphi_k \quad P}{\neg \psi} \quad \xrightarrow{\text{induction hypothesis}} \quad \frac{\varphi'_1 \dots \varphi'_k \quad P'}{\neg \psi'}$$

Plug in here a proof of $\neg \psi \vdash \psi^n$ in IQC, which exists by the first bullet

$\vdash \psi^n$

So we can still view CQC as a fragment of IQC. But this fragment no longer coincides with the set of formulas equivalent to a negation; moreover, the translation must now be defined inductively; we can't simply add $\neg \Pi$ in front of a formula.

Corollary IQC is undecidable.

Proof If IQC were decidable, we could decide CQC.

Given φ :

- Compute φ'
- decide whether $\varphi' \in$ IQC
- return answer

But CQC is undecidable (Church's theorem). \square