

Class 6: Topological Semantics

Kripke semantics is not the only semantics for IPC. Two other semantics which play an important role in studying IPC are the topological semantics (Tarski, ~1930) and the algebraic semantics based on the notion of a Heyting algebra. These different semantics can be ordered in terms of generality as follows:

Kripke \subseteq Topological \subseteq Algebraic

What this means is that every Kripke model can be seen as a special case of topological model, and every topological model can be seen as a special case of an algebraic model.

Let us first focus on topological semantics. We start by reviewing the counterpart for CPC.

CPC: possible world semantics

Def A possible world model for CPC is a pair $T = \langle W, V \rangle$ where W is a set and $V: P \rightarrow \wp(W)$.

Def The truth-set of φ in T is defined by:

- $[\varphi]_T = V(\varphi)$
- $[\neg\varphi]_T = \emptyset$
- $[\varphi \wedge \psi]_T = [\varphi]_T \cap [\psi]_T$
- $[\varphi \vee \psi]_T = [\varphi]_T \cup [\psi]_T$
- $[\varphi \rightarrow \psi]_T = \overline{[\varphi]_T} \cup [\psi]_T$
- $[\neg\neg\varphi]_T = \overline{\overline{[\varphi]_T}}$

Theor $\varphi \in \text{CPC} \Leftrightarrow$ for every possible world model $T = \langle W, V \rangle$, $[\varphi]_T = W$.

IPC: topological semantics

Def A topological space is a pair $S = \langle W, \tau \rangle$ where $\tau \subseteq \wp(W)$ such that:

1. $W, \emptyset \in \tau$
2. $X_1, \dots, X_n \in \tau \Rightarrow X_1 \cap \dots \cap X_n \in \tau$
3. $\mathbb{X} \subseteq \tau \Rightarrow \bigcup \mathbb{X} \in \tau$

A set $X \in \tau$ is called an open set.

Complements of open sets are called closed sets.

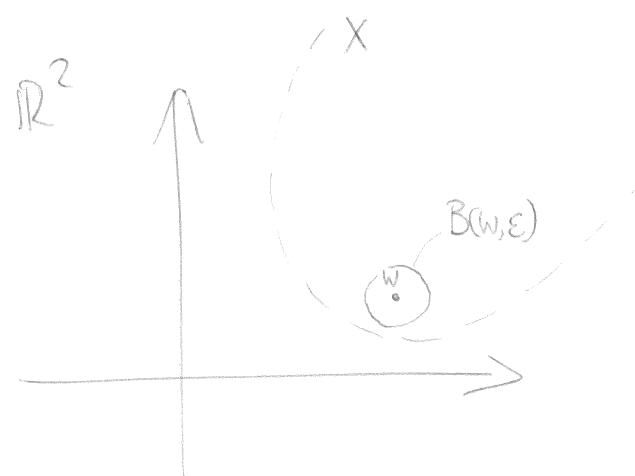
Ex. For any W we have the following topologies:

- $\tau = \{W, \emptyset\}$, the trivial topology
- $\tau = \wp(W)$, the discrete topology

Ex. Euclidean topology on \mathbb{R}^n .

$$W = \mathbb{R}^n$$

$$\tau = \{X \subseteq \mathbb{R}^n \mid \forall w \in X \exists \epsilon > 0 \text{ s.t. } B(w, \epsilon) \subseteq X\}$$



where $B(w, \epsilon)$ is the open ball with radius ϵ around w , i.e., the set of points whose distance from w is $< \epsilon$.

Def The interior of $X \subseteq W$ is $X^{\text{int}} = \bigcup \{Y \in \tau \mid Y \subseteq X\}$

Rehm

- $X^{\text{int}} \in \tau$
- $X^{\text{int}} \subseteq X$
- if $Y \in \tau$ and $Y \subseteq X$ then $Y \subseteq X^{\text{int}}$
- $X \in \tau \Leftrightarrow X = X^{\text{int}}$

Def A topological model is a triple $T = \langle W, \tau, V \rangle$ where $\langle W, \tau \rangle$ is a topological space and $V: P \rightarrow \tau$.

Def The truth-set of φ in a topological model T is defined as follows:

- $[\varphi]_T = V(\varphi)$
- $[\perp]_T = \emptyset$
- $[\varphi \wedge \psi]_T = [\varphi]_T \cap [\psi]_T$
- $[\varphi \vee \psi]_T = [\varphi]_T \cup [\psi]_T$
- $[\varphi \rightarrow \psi]_T = (\overline{[\varphi]}_T \cup [\psi]_T)^{\text{int}}$
- $[\neg \varphi]_T = (\overline{[\varphi]}_T)^{\text{int}}$

Theorem (Soundness and completeness of IPC for the topological interpretation)

$\varphi \in \text{IPC} \Leftrightarrow$ for every topological model T , $[\varphi]_T = W$.

Before proving this we give some examples.

Ex. \mathbb{R} with Euclidean topology

$$[\varphi] = \mathbb{R}^+$$



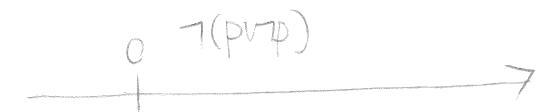
$$[\neg \varphi] = \mathbb{R}^-$$



$$\begin{aligned} [\varphi \vee \neg \varphi] &= [\mathbb{R}^+] \cup [\mathbb{R}^-] \\ &= \mathbb{R} - \{0\} \end{aligned}$$

since $[\varphi \vee \neg \varphi] \neq \mathbb{R}$, we know that $\varphi \vee \neg \varphi \notin \text{IPC}$.

$$\begin{aligned} [\neg(\varphi \vee \neg \varphi)] &= (\overline{\mathbb{R} - \{0\}})^{\text{int}} \\ &= \{0\}^{\text{int}} = \emptyset \end{aligned}$$



$$[\neg \neg(\varphi \vee \neg \varphi)] = (\overline{\emptyset})^{\text{int}} = \mathbb{R}$$



as expected, since $\neg \neg(\varphi \vee \neg \varphi) \in \text{IPC}$.

Interlude: interpreting the top. semantics

Def An observation space is a pair $\langle W, O \rangle$ with $O \subseteq \wp(W)$. We call the sets $X \in O$ observables.

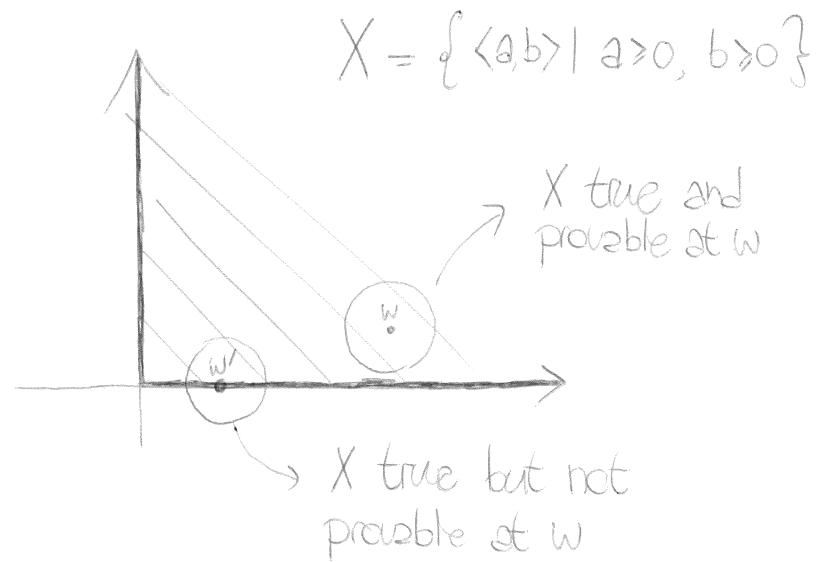
Ex. Imagine that we can only measure the position of a point in \mathbb{R}^2 with some approximation, however small. This corresponds to the following space:

$$W = \mathbb{R}^2 \quad O = \{B(w, \varepsilon) \mid w \in \mathbb{R}^2, \varepsilon > 0\}$$

Def In a space $\langle W, O \rangle$ we say that:

- X is true at $w \Leftrightarrow w \in X$
- X is provable at $w \Leftrightarrow \exists Y_1, \dots, Y_n \in O$
s.t. $w \in Y_1 \cap \dots \cap Y_n \subseteq X$

Idea: X is provable at w if there are a finite number of observables true at w s.t., after we have made those observations we will know that X is true.



Def We say that a set X is verifiable if $\forall w \in W: X \text{ is true at } w \Leftrightarrow X \text{ is provable at } w$

In words: verifiable sets are those for which truth and provability coincide.

The intuitionistic assumption that a sentence is true iff it is provable can be interpreted as a constraint that the proposition $[\varphi]$ expressed by φ should be a verifiable set.

Prop Given an observation space $\langle W, \mathcal{O} \rangle$, the verifiable sets form a topology τ on W . Conversely, every topology is the set of verifiable sets of some observation space.

The proposition guarantees that we can always think of open sets as propositions for which truth and provability coincide.

Going back to the top. model above:

- $\neg p$ is interpreted as the set of points where p is not just false but demonstrably false. This excludes o , where p is false but not demonstrably false.
- pvp is not universally true, since there are points, namely o , where p can be neither proved nor disproved.
- However, pvp is contradictory since pvp can never be disproved with finitely many observables.

- end of interlude -

Let $M = \langle W, R, V \rangle$ be an int. Kripke model. Let τ_R be the set of up-sets in M , i.e.

$$\tau_R = \{ X \subseteq W \mid \forall w, v: w \in X \wedge w R v \Rightarrow v \in X \}$$

Then one can check that:

- τ_R is a topology on W
- $T_M = \langle W, \tau_R, V \rangle$ is a topological model
(by persistency $V(p)$ is an up-set, so $V(p) \in \tau_R$)
- for every $w \in W$, $R[w]$ is the least open set containing w .

Prop For every i.k.m. M , world w , formula φ :

$$M, w \models \varphi \Leftrightarrow w \in [\varphi]_{T_M}$$

Proof By induction on φ .

Atoms, \perp , \wedge , \vee are all straightforward.

We consider $\varphi = \psi \rightarrow x$.

\Rightarrow Suppose $w \Vdash \psi \rightarrow x$.

This means that $\forall v \in R(w) : v \Vdash \psi \text{ or } v \Vdash x$

By ind. hypothesis: $\forall v \in R(w) : v \in \overline{[\psi]} \cup [x]$.

Hence, $R(w) \subseteq \overline{[\psi]} \cup [x]$.

Since $R(w) \in T_R$ we have $R(w) \subseteq (\overline{[\psi]} \cup [x])^{\text{int}}$

Hence, $w \in [\psi \rightarrow x]$. $\quad \square$

\Leftarrow Suppose $w \in [\psi \rightarrow x] = (\overline{[\psi]} \cup [x])^{\text{int}}$.

Since $R(w)$ is the least open set containing w ,

$R(w) \subseteq (\overline{[\psi]} \cup [x])^{\text{int}} \subseteq \overline{[\psi]} \cup [x]$.

Thus $\forall v \in R(w) : v \notin [\psi] \text{ or } v \in [x]$.

By IH this means: $\forall v \in R(w) : v \Vdash \psi \text{ or } v \Vdash x$.

Hence, $w \Vdash \psi \rightarrow x$. \square

Thus, Kripke semantics can be seen as a special case of topological semantics. Given a Kripke model M , we can regard it as a special kind of topological model ~~with~~ T_M with topology T_R induced by R . Then the interpretation of formulae in M is obtained as a special case of the topological semantics on T_M . Thus, we can view topological semantics as being at least as general as Kripke semantics (as a matter of fact, it is strictly more general, since many topological models are not induced by any Kripke models; only so-called Alexandroff topological spaces are generated in this way).

Kripke semantics \subseteq Topol. semantics

We are now equipped to show that IPC is sound and complete w.r.t. the top. sem.

Theor $\varphi \in \text{IPC} \Leftrightarrow$ for every top. model T ,
 $[\varphi]_T = W$

Proof

\Rightarrow Take a Hilbert-style axiomatization of IPC. It suffices to check that all axioms are top. valid and modus ponens preserves top. validity.

As an illustration, let's show that MP preserves top. validity.

Suppose $\varphi, \varphi \rightarrow \psi$ are top. valid, and take a top. model T . Then $[\varphi]_T = W$ and $W = [\varphi \rightarrow \psi]_T = (\overline{[\varphi]}_T \cup [\psi]_T)^{\text{int}} = (\overline{W} \cup [\psi]_T)^{\text{int}} = [\psi]_T = [\psi]_T$.

Where the last step is because $[\psi]_T$ is an open set, and the interior of an open set X is X .

Thus, $[\psi]_T = W$ for every T , which means that ψ is topologically valid.

\Leftarrow Suppose $\varphi \notin \text{IPC}$. Then, by the completeness of IPC for Kripke semantics, there's an i.k.m. M and a point w with $M, w \not\models \varphi$. By the previous proposition we have $w \notin [\varphi]_{T_M}$, and thus $[\varphi]_{T_M} \neq W$. This shows that φ is not topologically valid. \square