

Class 7: Algebraic Semantics

Overview

Kripke Sem \subseteq Topol. Sem \subseteq Algebraic Sem.

The general idea of algebraic semantics is to consider the propositions expressed by sentences as points in an algebra, and to view connectives as expressing operations in this algebra. In the case of intuitionistic logic, the central notion is that of a Heyting algebra, which can be introduced in two equivalent ways, as partially ordered sets, or as algebras, i.e., sets with a number of operations on them.

Heyting algebras

Posets
 $\langle A, \leq \rangle$

\approx

Algebras
 $\langle A, \perp, \top, \wedge, \vee, \rightarrow \rangle$

We go for the poset presentation here.

Def A partial order $\langle A, \leq \rangle$ is a Heyting algebra in case:

1. There are a least and a greatest element, \perp and \top
2. $\forall a, b \in A$ there exists a meet of a and b , i.e., an element $a \wedge b \in A$ s.t.
 - $a \wedge b \leq a$, $a \wedge b \leq b$
 - $\forall c \in A: c \leq a \wedge b \Rightarrow c \leq a \wedge b$

$\} \forall c \in A:$

$[c \leq a \wedge c \leq b \Rightarrow c \leq a \wedge b]$

3. $\forall a, b \in A$ there exists a join of a and b ,
i.e., an element $avb \in A$ such that:

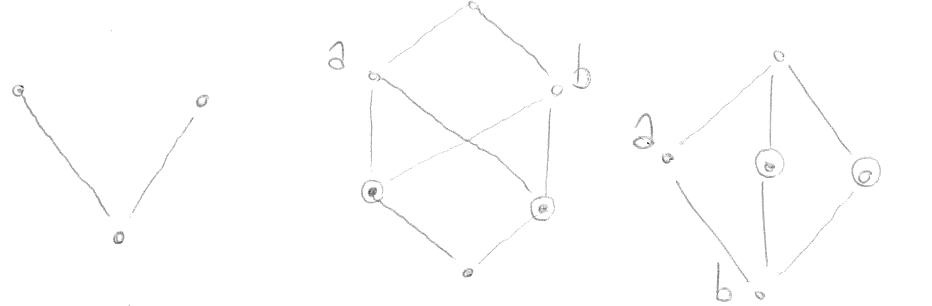
- $a \leq avb, b \leq avb$
 - $\forall c \in A: a \leq c \& b \leq c \Rightarrow avb \leq c$
- $\left. \begin{array}{l} \forall c \in A: \\ (a \leq c \& b \leq c) \end{array} \right\} avb \leq c$

4. $\forall a, b \in A$ there exists an implication of a and b ,
i.e., an element $a \rightarrow b$ s.t.

- $a \wedge (a \rightarrow b) \leq b$
 - $\forall c: a \wedge c \leq b \Rightarrow c \leq a \rightarrow b$
- $\left. \begin{array}{l} \forall c \in A: \\ a \wedge c \leq b \Leftrightarrow c \leq a \rightarrow b \end{array} \right\}$

We write a^* for $a \rightarrow \perp$ and call a^* the
pseudo-complement of a .

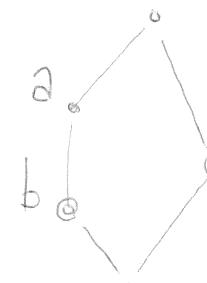
If $\forall a \in A$ we have $ava^* = \top$ we say that
 $\langle A, \leq \rangle$ is a Boolean algebra.



no T element

no meet of
a and b

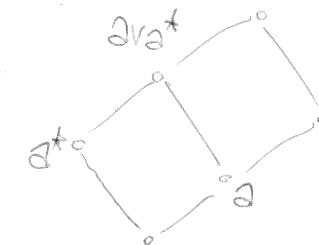
no implication of
a and b



no implication
of a and b



a HA which
is also a BA



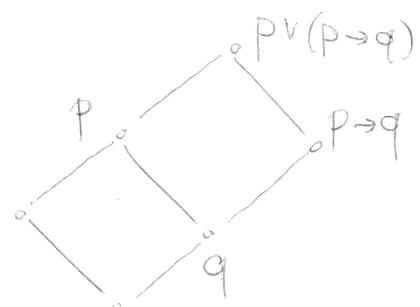
a HA which is
not a BA

Def An HA-model is a triple $A = \langle A, \leq, V \rangle$
where $\langle A, \leq \rangle$ is a Heyting algebra, $V: P \rightarrow A$.
Given such a model, we define the interpretation $[\cdot]_A: L \rightarrow A$ as follows

$$[p]_A = V(p) \quad [\perp]_A = \perp_A$$

$$[\varphi \circ \psi]_A = [\varphi]_A \circ [\psi]_A \quad \text{for } \circ \in \{\wedge, \vee, \rightarrow\}$$

Example



Def We say that φ is algebraically valid, $\models_{\text{HA}} \varphi$, in case for every HA model A we have $[\varphi]_A = T_A$.

$$[\varphi]_A = T_A.$$

Theor $\models_{\text{HA}} \varphi \Leftrightarrow \vdash_{\text{IPC}} \varphi$

As usual, for the soundness direction we can just take, e.g., a Hilbert-style system for IPC and show that axioms are HA-valid and mod.泡氏 preserves validity.

For completeness, we can proceed in two ways. One is to notice that every topological model (thus, also every Kripke model) can be seen as a special case of an algebraic model. Then completeness is inherited from the completeness for Kripke semantics.

Prop If $\langle W, \tau \rangle$ is a topological space, then $\langle \tau, \leq \rangle$ is a Heyting algebra with:

- $T = W$
- $X \wedge Y = X \cap Y$
- $X \rightarrow Y = (\overline{X} \cup Y)^{\text{int}}$
- $\perp = \emptyset$
- $X \vee Y = X \cup Y$

In particular, if $\mathcal{F} = \langle W, R \rangle$ is a Kripke frame, then $\langle \text{Up}(\mathcal{F}), \leq \rangle$ is a Heyting algebra.

Proof As on ~~the~~ illustration, we show that $(X \vee y)^{\text{int}}$ is the algebraic implication of X and y . We need to show:

- i. $X \cap (\bar{X} \vee y)^{\text{int}} \subseteq y$
- ii. $\forall z \in T: X \cap \bar{z} \leq y \Rightarrow z \leq (\bar{X} \vee y)^{\text{int}}$

For i, we have:

$$\begin{aligned} X \cap (\bar{X} \vee y)^{\text{int}} &\subseteq X \cap (\bar{X} \vee y) \\ &= (X \cap \bar{X}) \cup (X \cap y) \\ &= X \cap y \subseteq y \end{aligned}$$

For ii, we have:

$$X \cap z \leq y \Rightarrow z \leq \bar{X} \vee y$$

Since $(\bar{X} \vee y)^{\text{int}}$ is the largest open set contained in $\bar{X} \vee y$, it follows that $z \leq (\bar{X} \vee y)^{\text{int}}$.

□

Corollary If $T = \langle W, \tau, V \rangle$ is a topological model, then $A_T = \langle \tau, \leq, V \rangle$ is an HA-model, and $\text{tp}: [\varphi]_{A_T} = [\varphi]_T$.

Proof. This is immediate if we notice that the HA operations for A_T are exactly the operations used to interpret connectives in T . □

So, from the completeness of IPC for topological semantics we obtain completeness for HA-semantics. However, we can also proceed more directly, by viewing the logic IPC itself as an algebra, where eq. classes of formulas are ordered by entailment. This is known as the Lindenbaum-Tarski algebra of the logic. The idea can be applied to any logic, but I spell it out here for IPC.

Def $\varphi \equiv_{IPC} \psi \Leftrightarrow \varphi \vdash_{IPC} \psi \wedge \psi \vdash_{IPC} \varphi$

Def $\bar{\varphi}^{IPC} := \{\psi \mid \varphi \equiv_{IPC} \psi\}$

Def The Lindenbaum-Tarski algebra of IPC is the algebra $LT_{IPC} = \langle A_{IPC}, \leq_{IPC} \rangle$:

$$\cdot A_{IPC} = \{\bar{\varphi}^{IPC} \mid \varphi \in L_P\}$$

$$\cdot \bar{\varphi}^{IPC} \leq \bar{\psi}^{IPC} \Leftrightarrow \varphi \vdash_{IPC} \psi$$

Notice that the relation \leq_{IPC} is well-defined, since whether $\varphi \vdash_{IPC} \psi$ holds does not depend on the choice of the representatives φ and ψ .

Prop LT_{IPC} is a Heyting algebra with:

$$T_G = \bar{T}^{IPC} \quad \bar{\varphi}^{IPC} \wedge \bar{\psi}^{IPC} = \overline{\varphi \wedge \psi}^{IPC}$$

$$\perp_G = \bar{I}^{IPC} \quad \bar{\varphi}^{IPC} \vee \bar{\psi}^{IPC} = \overline{\varphi \vee \psi}^{IPC}$$

$$\bar{\varphi}^{IPC} \rightarrow \bar{\psi}^{IPC} = \overline{\varphi \rightarrow \psi}^{IPC}$$

Proof Straightforward using properties of IPC.

E.g. to show that $\overline{\varphi \wedge \psi}^{IPC}$ is the meet of $\bar{\varphi}^{IPC}$ and $\bar{\psi}^{IPC}$ we need to show:

$$\text{i. } \overline{\varphi \wedge \psi}^{IPC} \leq \bar{\varphi}^{IPC}, \quad \overline{\varphi \wedge \psi}^{IPC} \leq \bar{\psi}^{IPC}$$

$$\text{ii. } \forall \bar{x}^{IPC}: \bar{x}^{IPC} \leq \bar{\varphi}^{IPC} \wedge \bar{x}^{IPC} \leq \bar{\psi}^{IPC} \Rightarrow \bar{x}^{IPC} \leq \overline{\varphi \wedge \psi}^{IPC}$$

By the definition of the ordering, this means:

$$\text{i. } \varphi \wedge \psi \vdash_{IPC} \varphi, \quad \varphi \wedge \psi \vdash_{IPC} \psi$$

$$\text{ii. } \forall x: x \vdash_{IPC} \varphi \wedge x \vdash_{IPC} \psi \Rightarrow x \vdash_{IPC} \varphi \wedge \psi$$

Part i is clear by the (ne)-rule, part ii by (ni). \square

Def The canonical HA model for IPC is

$$A_{IPC} = \langle A_{IPC}, \leq_{IPC}, V_{IPC} \rangle \text{ where } V_{IPC}(p) = \bar{P}^{IPC}.$$

Prop $\vdash_{\text{IPC}} \varphi : [\varphi]_{A_{\text{IPC}}} = \bar{\varphi}^{\text{IPC}}$

Proof By a straightforward induction on φ :

- Atoms: $[\varphi]_{A_{\text{IPC}}} = V(\varphi) = \bar{\varphi}^{\text{IPC}}$
- Conjunction: $[(\varphi \wedge \psi)]_{A_{\text{IPC}}} = [\varphi]_{A_{\text{IPC}}} \wedge [\psi]_{A_{\text{IPC}}}$
 $\stackrel{\text{IH}}{=} \bar{\varphi}^{\text{IPC}} \wedge \bar{\psi}^{\text{IPC}}$
 $= \overline{\varphi \wedge \psi}^{\text{IPC}}$

Etc.

□

Now we can prove the completeness of IPC
for the algebraic semantics directly.

Proof Suppose $\vdash_{\text{HA}} \varphi$. Then in particular we must have $[\varphi]_{A_{\text{IPC}}} = T_{A_{\text{IPC}}} = \top^{\text{IPC}}$.

But also $[\varphi]_{A_{\text{IPC}}} = \bar{\varphi}^{\text{IPC}}$. So $\bar{\varphi}^{\text{IPC}} = \top^{\text{IPC}}$,
i.e., $\varphi \models_{\text{IPC}} T$, so $\vdash_{\text{IPC}} \varphi$. □