

## Class 5: IPC, CPC, and modal logic

• CPC is an extension of IPC:

$$\text{CPC} = \text{IPC} + \neg\neg\varphi \rightarrow \varphi = \text{IPC} + \varphi \vee \neg\varphi$$

• Today we'll see that CPC can also be regarded as a fragment of IPC via a translation due to Gödel and Gentzen.

Def (Translation between logics)

Let  $L, L'$  be two logics with languages  $\mathcal{L}, \mathcal{L}'$ . A translation of  $L$  to  $L'$  is a map  $(\cdot)^*: \mathcal{L} \rightarrow \mathcal{L}'$  such that for all  $\varphi_1, \dots, \varphi_k, \psi \in \mathcal{L}$ :

$$(\varphi_1, \dots, \varphi_k \vdash_L \psi \Leftrightarrow (\varphi_1^*, \dots, \varphi_k^* \vdash_{L'} \psi^*)$$

Theor (Glivenko)  $\varphi \in \text{CPC} \Leftrightarrow \neg\neg\varphi \in \text{IPC}$

Lemma Let  $M$  be a finite intuit. Kripke model.  $M, w \Vdash \neg\neg\varphi \Leftrightarrow$  for all endpoints  $e \in R(w)$ :  $M, e \Vdash \varphi$

Proof

Recall from class 2 that endpoints behave like valuations in classical logic, and thus obey CPC.

Suppose  $M, w \Vdash \neg\neg\varphi$ . Take any endpoint  $e \in R(w)$ . By persistency,  $M, e \Vdash \neg\varphi$ , and since endpoints behave classically also  $M, e \Vdash \varphi$ .

Conversely suppose  $M, w \not\Vdash \neg\neg\varphi$ . Then  $\exists e \in R(w)$  s.t.  $M, e \Vdash \neg\varphi$ . Let  $e$  be an endpoint in  $R(w)$ , which must exist because  $M$  is finite. By persistency  $M, e \Vdash \neg\varphi$ , so also  $M, e \Vdash \varphi$ . And by the transitivity of  $R$  we have  $e \in R(w)$ .  $\square$

## Proof of Glivenko's theorem

$\Leftarrow$  Suppose  $\pi\varphi \in \text{IPC}$ . Since  $\text{IPC} \subseteq \text{CPC}$  we have  $\pi\varphi \in \text{CPC}$ . Since  $\pi\varphi \equiv_{\text{CPC}} \varphi$ , also  $\varphi \in \text{CPC}$ .

$\Rightarrow$  Suppose  $\pi\varphi \notin \text{IPC}$ . By the finite model property of IPC we have a finite M and a point w s.t.  $M, w \not\models \pi\varphi$ . By the previous lemma we have an endpoint e  $\in R(w)$  s.t.  $M, e \models \varphi$ . Let  $v_e$  be the valuation function associated to e:

$$v_e(p) = \begin{cases} 1 & \text{if } e \in V(p) \\ 0 & \text{if } e \notin V(p) \end{cases}$$

We know that  $M, e \models \varphi \Leftrightarrow v_e(\varphi) = 1$ .

Since  $M, e \not\models \pi\varphi$  we have  $v_e(\varphi) = 0$ , so  $\varphi \notin \text{CPC}$ .  $\square$

Thus, e.g.,  $\pi(\varphi \vee p) \in \text{IPC}$ ,  $\pi(\pi p \rightarrow p) \in \text{IPC}$ .

Def (Negative translation of CPC into IPC, aka Gödel-Gentzen translation)

- $p^n = \pi p$
- $(\varphi \rightarrow \psi)^n = \varphi^n \rightarrow \psi^n$
- $(\varphi \vee \psi)^n = \pi(\varphi^n \vee \psi^n)$
- $(\varphi \wedge \psi)^n = \varphi^n \wedge \psi^n$

Remark For all  $\varphi$ ,  $\varphi \equiv_{\text{CPC}} \varphi^n$

Lemma For all  $\varphi$ ,  $\varphi^n \equiv_{\text{IPC}} \pi\varphi^n$

Proof Induction on  $\varphi$ .

Recall from the exercise session:

- (1)  $\pi(\varphi \wedge \psi) \equiv_{\text{IPC}} \pi\varphi \wedge \pi\psi$
- (2)  $\pi(\varphi \rightarrow \psi) \equiv_{\text{IPC}} \pi\varphi \rightarrow \pi\psi$

i. Atoms:  $p^n = \pi p \equiv_{\text{IPC}} \pi \pi p = \pi p^n$

$$\text{ii } \perp. \quad \perp^n = \perp \equiv_{\text{IPC}} \top \perp = \top \perp^n$$

$$\text{iii } \Delta. \quad (\varphi \wedge \psi)^n \stackrel{\text{IH}}{=} \varphi^n \wedge \psi^n \equiv_{\text{IPC}} \top \varphi^n \wedge \top \psi^n \\ \stackrel{(1)}{=} \top (\varphi^n \wedge \psi^n) = \top (\varphi \wedge \psi)^n$$

iv  $\exists$ . Analogous to previous case, using (2).

$$\text{v } \vee. \quad (\varphi \vee \psi)^n = \top (\varphi^n \vee \psi^n) \equiv_{\text{IPC}} \top \top (\varphi^n \vee \psi^n) \\ = \top (\varphi \vee \psi)^n. \quad \square$$

Theorem  $((\cdot))^n$  is a translation of CPC into IPC

For all  $\varphi_1, \dots, \varphi_k, \psi$ :

$$\varphi_1, \dots, \varphi_k, \vdash_{\text{CPC}} \psi \Leftrightarrow \varphi_1^n, \dots, \varphi_k^n, \vdash_{\text{IPC}} \psi^n$$

Proof by Remark above

$$\varphi_1, \dots, \varphi_k, \vdash_{\text{CPC}} \psi \Leftrightarrow \varphi_1^n, \dots, \varphi_k^n, \vdash_{\text{IPC}} \psi^n$$

$$\Leftrightarrow \vdash_{\text{IPC}} \varphi_1^n \wedge \dots \wedge \varphi_k^n \rightarrow \psi^n$$

Givens

$$\Leftrightarrow \vdash_{\text{IPC}} \top (\varphi_1^n \wedge \dots \wedge \varphi_k^n \rightarrow \psi^n)$$

(1), (2)

$$\Leftrightarrow \vdash_{\text{IPC}} \top \varphi_1^n \wedge \dots \wedge \top \varphi_k^n \rightarrow \top \psi^n$$

Lemma

$$\Leftrightarrow \vdash_{\text{IPC}} \varphi_1^n \wedge \dots \wedge \varphi_k^n \rightarrow \psi^n$$

$$\Leftrightarrow \varphi_1^n, \dots, \varphi_k^n, \vdash_{\text{IPC}} \psi^n \quad \square$$

So CPC can be viewed as a fragment of IPC  
modulo identifying each formula  $\varphi$  with its transl.  $\varphi^n$ .

While classical propositional logic can be seen as a fragment of int. prop. logic, the latter can in turn be seen as a fragment of certain systems of classical model logic.

We will focus on the modal logic S4, though similar translations are possible for other logics.



Def A classical Kripke model is a triple  $M = \langle W, R, V \rangle$  where

- $W$  is a set
- $R \subseteq W \times W$  is a binary relation on  $W$
- $V: P \rightarrow \wp(W)$  is a valuation function

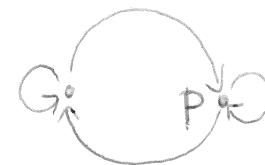
Def  $M$  is an S4-model if  $R$  is

- reflexive:  $wRw$
- transitive:  $wRv, vRu \Rightarrow wRu$

Ram Intuitionistic Kripke models are S4 models, but not vice versa:

- $R$  need not be anti-symmetric
- $V$  need not be persistent

Example The following is an S4 model but not an i.Km.:



Def The language of modal logic is given by:

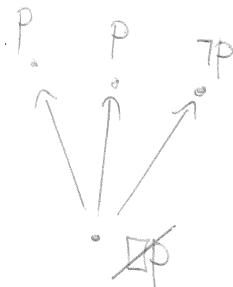
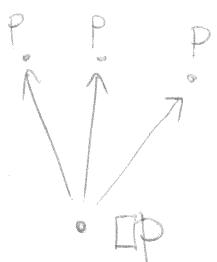
$$\varphi ::= p \mid \perp \mid \varphi_1 \varphi_2 \mid \varphi \rightarrow \varphi \mid \varphi \vee \varphi \mid \Box \varphi$$

## Def (Semantics of modal logic)

let  $M$  be a Kripke model,  $w$  a point.

- $M, w \models p \Leftrightarrow w \in V(p)$
- Connectives have the classical clauses (including  $\rightarrow$ , interpreted materially)
- $M, w \models \Box\varphi \Leftrightarrow \forall v \in R(w): M, v \models \varphi$

## Examples



## Def (Modal logic S4)

S4 is the modal logic defined by the following Hilbert-style proof system:

### Axioms

- All instances of CPC tautologies
- K:  $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
- T:  $\Box\varphi \rightarrow \varphi$  (reflexivity)
- 4:  $\Box\varphi \rightarrow \Box\Box\varphi$  (transitivity)

### Rules

- Modus ponens:  $\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}$
- Necessitation:  $\frac{\varphi}{\Box\varphi}$

Write  $\varphi_1, \dots, \varphi_n \vdash \psi$  if  $\varphi_1 \wedge \dots \wedge \varphi_n \rightarrow \psi$  is derivable in this system.

The following is a standard model logic theorem:

Theorem: S4 is sound and complete with respect to S4 models. That is, for all modal formulas  $\varphi_1 \rightarrow \varphi_n \psi$ , the following are equivalent:

(i)  $\varphi_1 \rightarrow \varphi_n \vdash \psi$

(ii) for all S4-models M and points w,  
if  $M, w \models \varphi_i$  for  $i = 1, \dots, n$  then  $M, w \models \psi$ .

Def (Model) translation of IPC

The translation  $(\cdot)^{\square}$  from the language of IPC to that of model logic is defined inductively as follows:

- $p^{\square} = \Box p$
- $\perp^{\square} = \perp$
- $(\varphi \wedge \psi)^{\square} = \varphi^{\square} \wedge \psi^{\square}$
- $(\varphi \vee \psi)^{\square} = \varphi^{\square} \vee \psi^{\square}$
- $(\varphi \rightarrow \psi)^{\square} = \Box(\varphi^{\square} \rightarrow \psi^{\square})$

Examples

- $(p \rightarrow q)^{\square} = \Box(p^{\square} \rightarrow q^{\square}) = \Box(\Box p \rightarrow \Box q)$
- $(p \wedge q)^{\square} = p^{\square} \wedge q^{\square} = \Box p \wedge \Box q$
- $(\neg p)^{\square} = (p \rightarrow \perp)^{\square} = \Box(p^{\square} \rightarrow \perp^{\square})$   
 $= \Box(\Box p \rightarrow \perp) = \Box \perp \Box p$

Rem A possible interpretation of  $\Box$ , intuitively, is as 'it is provable that'. With this reading in mind the model translation can be seen as a way of making explicit in a classical setting the provability interpretation implicit in the intuitionistic way of reading formulas.

Thus, e.g.,  $(p \vee \neg p)^{\square} = \Box p \vee \Box \neg p$ , which brings out that, in IPC,  $p \vee \neg p$  is read as: either we can prove p, or we can prove that there is no proof of p.

Lemma Let  $M$  be an int. Kripke model.

Then  $M$  is also an S4 model, and for all points  $w$  and prop. formulas  $\varphi$ :

$$M, w \Vdash \varphi \Leftrightarrow M, w \models \varphi^\square$$

intuit. Kripke semantics      modal logic

Proof By induction on  $\varphi$ .

$$\text{Atoms } M, w \Vdash p \Leftrightarrow w \in V(p)$$

$$\Leftrightarrow R[w] \subseteq V(p)$$

$$\xleftarrow{\text{persistency of } V} \Leftrightarrow \forall v \in R[w]: M, v \Vdash p$$

$$\Leftrightarrow M, w \models \Box p$$

L, \wedge, V: straightforward

$$\Rightarrow M, w \Vdash \varphi \rightarrow \psi \Leftrightarrow \forall v \in R[w]: M, v \Vdash \varphi \Rightarrow M, v \Vdash \psi$$

$$\Leftrightarrow \forall v \in R[w]: M, v \models \varphi^\square \Rightarrow M, v \models \psi^\square$$

$$\Leftrightarrow \forall v \in R[w]: M, v \models \varphi^\square \rightarrow \psi^\square$$

$$\Leftrightarrow M, w \models \Box(\underbrace{\varphi^\square \rightarrow \psi^\square}_{(\varphi \rightarrow \psi)^\square}) \quad \square$$

Theor  $((\cdot)^\square)$  is a translation of IPC to S4

For all prop. formulas  $\varphi_1, \dots, \varphi_n, \psi$ :

$$\varphi_1, \dots, \varphi_n \vdash_{IPC} \psi \Leftrightarrow \varphi_1^\square, \dots, \varphi_n^\square \vdash_{S4} \psi^\square$$

Proof

$\Leftarrow$  Suppose  $\varphi_1, \dots, \varphi_n \vdash_{IPC} \psi$ .

Then there's an i.k.m.  $M$  and point  $w$  s.t.

$M, w \Vdash \varphi_i$  for  $i=1, \dots, n$  but  $M, w \not\Vdash \psi$ .

By the previous lemma we have

$M, w \models \varphi_i^\square$  for  $i=1, \dots, n$  but  $M, w \not\models \psi^\square$ .

Since  $M$  is an S4-model,  $\varphi_1^\square, \dots, \varphi_n^\square \not\models_{S4} \psi^\square$ .

$\Rightarrow$  This direction is slightly more tricky, since not every S4-model is an i.k.m. suitable for intuitionistic Kripke semantics.

We will look at this direction in the next exercise session.