Intuitionistic conditional logics

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Abstract

Building on recent work by Weiss (2019a,b), we study conditional logics in the intuitionistic setting. We consider a number of semantic conditions which give rise, among others, to intuitionistic counterparts of Lewis's logic VC and Stalnaker's C2. We show how to obtain a sound and complete axiomatization of each logic arising from a combination of these conditions. On the way, we remark how, in the intuitionistic setting, certain classically equivalent principles of conditional logic come apart, and how certain logical connections between different principles no longer hold.

Keywords: conditional logic · intuitionistic logic · variably strict conditionals · modal logic · non-monotonic reasoning

1 Introduction

Conditional logics arose from the work of Adams (1965), Stalnaker (1968) and Lewis (1973) on the semantics of conditionals, and their study soon grew into an active area of research in logic (see Nute, 1984, for an overview) with tight connections to modal logic (Chellas, 1975; Segerberg, 1989) and non-monotonic reasoning (Kraus *et al.*, 1990; Veltman, 1996). In this area, scholars have studied conservative extensions of classical logic with a conditional operator $>$ and classified a number of logics which result from imposing natural constraints on the associated models. Among these, a prominent place is played by the logics V, VW and VC, arising from Lewis's interpretation of conditionals in terms of *minimal change*. These logics arise from the idea that $\varphi > \psi$ is true in a world w if among the worlds where φ is true, the ones which are most similar to w are ψ-worlds. Another logic that plays an important role is Stalnaker's system C2, which results from the idea that the truth-value of $\varphi > \psi$ in a world w depends on the truth-value of ψ in a single φ-world—intuitively, the φ-world which differs minimally from w.

In the literature, conditional logics are studied as conservative extensions of classical logic. However, there seems to be no reason why the enterprise of modeling conditional reasoning should be confined to the classical setting. It seems quite interesting to consider how non-classical logics can be equipped with the tools to model conditional reasoning. Among non-classical logics, a prominent place is occupied by *intuitionistic logic*, where only constructive forms of reasoning are allowed. A natural question is, then: what do the intuitionistic counterparts of prominent conditional logics look like?

Recently, Weiss (2019a,b) took a first step in this direction, showing how to extend intuitionistic Kripke semantics with the structure needed to interpret a conditional operator $>$, and studying the intuitionistic counterparts of some weak conditional logics, including the minimal conditional logic `CK` of Chellas (1975). In this paper, we will follow up on his work, and study the intuitionistic counterparts of the most important conditional logics, including `V`, `VW`, `VC`, and `C2`. In fact, we will do something more general: we will consider a range of semantic conditions, and a corresponding range of axioms, and we will show that *any combination* of those axioms gives rise to a conditional logic which is sound and complete for the class of models satisfying the associated conditions.

In order to achieve this goal, we will propose a small modification of the semantics given by Weiss (2019a,b). This modification does not affect the generality of the semantics: every Weiss model can be turned into one of our models without affecting the satisfaction of formulas, and vice versa. However, the modified notion of models will facilitate a simple and elegant correspondence between semantic conditions and syntactic axioms, which seems hard to obtain otherwise.

Before getting down to business, let us mention two reasons why studying the logic of conditionals in an intuitionistic setting is an interesting enterprise.

First, intuitionistic logic is already equipped with its own conditional operator $\to$. Unlike the material conditional of classical logic, which is truth-functional, the conditional of intuitionistic logic is an intensional operator, and it is semantically very close to that of the operator $>$ in conditional logics: both conditionals check whether the consequent is satisfied everywhere within a certain set of possible worlds determined by the antecedent. The difference is which set of worlds is picked out by each of them. Crucially, $\to$ is constrained to quantifying over states of affairs which are *possible* from the perspective of the evaluation world, while $>$ is allowed to quantify over counterfactual states of affairs as well. Thus, a natural interpretation is to view $p \to q$ as standing for an epistemic conditional like (1-a), and to view $p > q$ as standing for an ontic conditional like (1-b).

(1) a. If the butler didn't do it, the gardener did.
    b. If the butler had not done it, the gardener would have.

In this sense, the logics that we will look at can be seen as logics in which two different kinds of conditionals, epistemic and ontic, interact with each other.

Second, studying the principles of conditional logic from the perspective of intuitionistic logic allows us to ask which of these principles only stem from assumptions about the semantics of conditionals, and which stem partly from the classicality of the underlying logic. As a concrete example, consider Stalnaker's conditional logic C2. Usually, this logic is characterized by the conditional excluded middle axiom, $(\varphi > \psi) \vee (\varphi > \neg\psi)$, which says that every antecedent yields either a sentence or its negation. Classically, this principle is equivalent to a determinacy principle stating that, whenever an antecedent yields a disjunction, it yields a specific one of the disjuncts: $(\varphi > \psi \vee \chi) \to (\varphi > \psi) \vee (\varphi > \chi)$. It turns out that, intuitionistically, the latter, and not the former, is the appropriate way to capture Stalnaker's constraint—the uniqueness assumption. Conditional excluded middle is a consequence of the uniqueness assumption—a proper conditional principle—paired with the bivalence of classical logic.[1]

The paper is structured as follows. In Section 2 we will define a semantics for intuitionistic logic extended with the operator $>$. In Section 3 we discuss the relations between this semantics and the one of Weiss (2019a,b). In Section 4 we will define a general notion of an intuitionistic conditional logic. In Section 5 we will describe how to construct a canonical model for a given intuitionistic conditional logic L, in which all L-invalid entailments can be falsified. In Section 6 we define a number of semantic conditions on intuitionistic conditional models, and discuss their significance. In Section 7 we introduce axioms corresponding to these conditions, and show that each of these axioms is valid on models satisfying the corresponding condition. Finally, in Section 8 we should that these axioms are *canonical* for the semantic conditions, in the following sense: if L contains one of these axioms, then the canonical model for L satisfies the corresponding semantic condition. Using this fact, we get a soundness and completeness result for a variety of intuitionistic conditional logics, including the intuitionistic counterparts of V, VW, VC, and C2. Section 9 summarizes our findings and outlines some directions for further work.

2 Intuitionistic conditional semantics

In this section we describe our semantics for intuitionistic propositional logic equipped with an extra conditional operator $>$. First, let us specify the language that we will work with.

Definition 2.1 (Language)**.**
The language $\mathcal{L}^>$ of intuitionistic propositional conditional logic based on a set $\mathcal{P}$ of atoms is given by the following BNF definition, where $p \in \mathcal{P}$:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \varphi > \varphi$$

As usual in intuitionistic logic, negation, the tautology symbol $\top$, and the biconditional are defined as follows:

[1] Weiss (2019a,b) ties Stalnaker's logic to conditional excluded middle, and goes on to claim that there is no intuitionistic counterpart of C2, since he shows that any intuitionistic conditional logic containing conditional excluded middle is actually classical. In fact, as we will see, there is a natural intuitionistic counterpart of C2: its axiomatization, however, does not involve conditional excluded middle.

- $\neg\varphi := \varphi \to \bot$
- $\top := \neg\bot$
- $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$

We will interpret our language with respect to models that extend intuitionistic Kripke models with the structure needed to interpret the operator $>$. First, let us recall the standard notion of an *intuitionistic Kripke frame*.

Definition 2.2 (Intuitionistic Kripke frames)**.**
An intuitionistic Kripke frame is a pair $\mathcal{F} = \langle W, \leq \rangle$, where:

- W is a set, whose elements are called *worlds*;
- $\leq$ is a partial order on W, the *refinement* ordering.

Given a frame $\langle W, \leq \rangle$, we denote by $\mathrm{Up}_{\leq}(W)$ denotes the set of up-sets of W:

$$\mathrm{Up}_{\leq}(W) = \{X \subseteq W \mid \forall w, v : w \in X \text{ and } w \leq v \text{ implies } v \in X\}$$

Moreover, if $w \in W$, we denote by $w^{\uparrow}$ the set of $\leq$-successors of w:

$$w^{\uparrow} := \{v \in W \mid w \leq v\}$$

We are now ready to define our intuitionistic conditional models. These are structures which enrich an intuitionistic Kripke model with a family R_X of binary relation indexed by propositions, where propositions are designated sets of possible worlds.

Definition 2.3 (Intuitionistic conditional models)**.**
An intuitionistic conditional model, abbreviated as ICM, is a structure $M = \langle W, \leq, \mathcal{A}, \{R_X \mid X \in \mathcal{A}\}, V \rangle$ where:

- $\langle W, \leq \rangle$ is an intuitionistic Kripke frame;
- $\mathcal{A} \subseteq \mathrm{Up}_{\leq}(W)$ is a set of up-sets called *propositions*.
- For each $X \in \mathcal{A}$, R_X is a binary relation on W; the set of R_X-successors of a world w is denoted by $R_X[w]$:

$$R_X[w] := \{v \in W \mid wR_Xv\}$$

- $V : \mathcal{P} \to \mathcal{A}$ is a *valuation function*, assigning to each atom a proposition.

We require the components of an ICM to satisfy certain conditions:

- Closure of $\mathcal{A}$ under logic: $\mathcal{A}$ contains $\emptyset$, it is closed under intersection, union, and the following operations:
  - $X, Y \mapsto \{w \in W \mid X \cap w^{\uparrow} \subseteq Y\}$

– $X, Y \mapsto \{w \in W \mid R_X[w] \subseteq Y\}$

- Upwards-closure of $R_X[w]$: for any $w \in W$, $R_X[w] \in \mathrm{Up}_{\leq}(W)$.
- Monotonicity of $R_X[\cdot]$: if $w \leq v$ then $R_X[w] \supseteq R_X[v]$.

One may think of worlds as partial states in a process of inquiry. In each world, a sentence φ may or may not have been established. The relation $w \leq v$ means that v is a refinement of w: that is, v establishes anything that w establishes, and possibly more. Alternatively, one may think of a world $v \geq w$ as one of the ways in which the process of inquiry may develop further from w. The function V maps p to the set of those worlds where it is established that p.

The intended interpretation of R_X is that the elements $v \in R_X[w]$ are those worlds which, from the standpoint of w, may have obtained if the proposition X had been established. We will refer to $R_X[w]$ as the *hypothetical context* generated at w by the making the ontic assumption that X.

The closure conditions on $\mathcal{A}$ are needed to ensure that, given the semantics below, the object $|\varphi|$ expressed by each sentence is always a proposition, and therefore that a corresponding relation $R_{|\varphi|}$ is available in the model.

The second condition requires $R_X[w]$ to be upwards-closed.[2] Conceptually, it can be motivated as follows: if $v \in R_X[w]$, that means that at w we consider it possible that, had X been the case, v may have obtained. Since v may evolve into any of its successors, each successor of v may have obtained if X had been the case. Therefore, each successor of v should be in $R_X[w]$.

Finally, the monotonicity condition says that, if $w \leq v$, then the hypothetical context $R_X[v]$ that we get at v is stronger than the context $R_X[w]$ that we get at w. This is a natural constraint: $w \leq v$ means that all the information available at w is also available at v; this includes conditional information about how the world would be like if X were the case; this means that any counterfactual possibility u which can be ruled out at w ($u \notin R_X[w]$) can also be ruled out at v ($u \notin R_X[v]$). This means that $R_X[v] \subseteq R_X[w]$.

Next, let us define a notion of satisfaction, capturing when a formula φ is established in a world w in an intuitionistic conditional model M.

Definition 2.4 (Semantics)**.**

1. $M, w \models p \iff w \in V(p)$
2. $M, w \not\models \bot$
3. $M, w \models \varphi \wedge \psi \iff M, w \models \varphi$ and $M, w \models \psi$
4. $M, w \models \varphi \vee \psi \iff M, w \models \varphi$ or $M, w \models \psi$
5. $M, w \models \varphi \rightarrow \psi \iff \forall v \geq w : M, v \models \varphi$ implies $M, v \models \psi$
6. $M, w \models \varphi > \psi \iff \forall v \in R_{|\varphi|}[w] : M, v \models \psi$

[2] This is the main difference between our semantics and the one of Weiss (2019a,b), who does not impose this constraint. We will discuss this in more detail in the next section.

Where the set $|\varphi|$, called the *proposition expressed* by of φ in M, is defined as:

- $|\varphi| := \{w \in W \mid M, w \models \varphi\}$

To simplyfy notation, in the following we will usually write R_φ instead of $R_{|\varphi|}$.

Clauses 1-5 are just the standard clauses of intuitionistic Kripke semantics. Clause 6 is the standard interpretation of the operator $>$ in conditional logic: $\varphi > \psi$ is satisfied at w if ψ is satisfied at all the R_φ-successors of w; in other words, $\varphi > \psi$ is satisfied at w iff ψ holds throughout the hypothetical context which results from making the ontic assumption that φ at w.

It is interesting to note that, although this is not obvious from the previous definition, the two conditionals can be seen as having essentially the same semantic clause. To see this, notice that we can think of those $\leq$-successors of w which satisfy φ as those states which may be the actual state if φ becomes established. We can then think that making the epistemic assumption that φ amounts to imagining that we are in one of these worlds. More technically, let us define a family $\{S_X \mid X \in \mathcal{A}\}$ as follows:

- $S_X[w] := X \cap w^{\uparrow}$

Let us write S_φ for $S_{|\varphi|}$. We can think of $S_\varphi[w]$ as the hypothetical context generated at w by supposing φ as an *epistemic*, rather than ontic, assumption. Then the semantics for the two conditionals can be made exactly parallel:

- $M, w \models \varphi \to \psi \iff S_\varphi[w] \subseteq |\psi|$
- $M, w \models \varphi > \psi \iff R_\varphi[w] \subseteq |\psi|$

The difference between the two conditionals lies in the fact that we have two modes of making an assumption φ: the epistemic mode, corresponding to $S_\varphi[\cdot]$, and the ontic mode, corresponding to $R_\varphi[\cdot]$.

As in standard intuitionistic Kripke semantics, the satisfaction relation is *persistent*: anything which is established in a world w remains established at any refinement of w.

Proposition 2.5 (Persistency)**.**
For every ICM M, if $w \leq v$ then $M, w \models \varphi$ implies $M, v \models \varphi$.

Proof. The proof is by induction on φ. We prove the case in which $\varphi = \psi > \chi$; the remaining cases are familiar from intuitionistic logic.

Suppose $w \leq v$ and $M, w \models \psi > \chi$. This means that $R_\psi[w] \subseteq |\chi|$. By the monotonicity of $R_\varphi[\cdot]$ we have $R_\psi[v] \subseteq R_\psi[w]$. Therefore also $R_\psi[v] \subseteq |\chi|$, which means that $M, v \models \psi > \chi$. □

It is interesting to note that, given an intuitionistic Kripke model, there are always two canonical ways to equip it with a family $\{R_X \mid X \in \text{Up}_{\leq}(W)\}$ so that the result is an ICM: one consists in taking $R_X[w]$ to be X; the other consists in taking $R_X[w]$ to be the set of refinements of w which are in X.

Definition 2.6. Let $M = \langle W, \leq, V \rangle$ be a standard intuitionistic Kripke model. We define two corresponding ICMs, M^u and M^i, by expanding M with the set of propositions $\mathcal{A} := \mathrm{Up}_{\leq}(W)$ and with the family $\{R_X \mid X \in \mathrm{Up}_{\leq}(W)\}$, where the relations R_φ are defined as follows:

- for M^u, define $R_X[w] := X$
- for M^i, define $R_X[w] := S_X[w] = X \cap w^{\uparrow}$

It is easy to see that both M^u and M^i are proper ICMs, that is, the conditions of Definition 2.3 are satisfied. As the following observation brings out, the model M^u yields a universal strict account of $>$, which looks at all possible worlds in the model, while M^i renders the operator $>$ identical to the intuitionistic conditional $\to$.

Observation 2.7. For any intuitionistic Kripke model M we have:

- $M^u, w \models \varphi > \psi \iff \forall v \in W : M^u, v \models \varphi \to \psi \iff |\varphi| \subseteq |\psi|$
- $M^i, w \models \varphi > \psi \iff M^i, w \models \varphi \to \psi$

It is also worth pointing out that the semantics for *classical* conditional logic can be retrieved as a special case. Indeed, the classical conditional models studied, e.g., by Segerberg (1989) correspond one-to-one to ICMs where the refinement relation $\leq$ is the identity relation; moreover, restricted to these models, our semantic clauses boil down to the classical ones. So, from a semantic perspective intuitionistic conditional logic is a generalization of classical conditional logic.

The notions of validity and entailment over a class of models are defined as usual.

Definition 2.8 (Validity, entailment)**.**

- We say that φ is valid in a model M if it is satisfied at every world in M.
- We say that φ is valid over a class of models $\mathcal{C}$, written $\models_{\mathcal{C}} \varphi$, if it is valid in every model $M \in \mathcal{C}$.
- We say that a set of formulas Φ entails a formula ψ over a class of models $\mathcal{C}$, notation $\Phi \models_{\mathcal{C}} \psi$, if for every $M \in \mathcal{C}$ and every w in M: if $M, w \models \varphi$ for all $\varphi \in \Phi$ then $M, w \models \psi$.

3 Relations with Weiss's semantics

The semantics in the previous section is very close to, and directly inspired by, the one recently proposed by Weiss (2019b) (§6). However, it differs from it in one respect: unlike Weiss, we require $R_X[w]$ to be an up-set. Other than the lack of this requirement, Weiss's notion of a model is exactly the same as ours, and formulas are interpreted by the same semantic clauses.[3]

In a sense, the difference is superficial: on the one hand, every model in our sense is also a model in the sense of Weiss (2019b), and in this class of models, the two semantics coincide. Conversely, there is a straightforward way of translating every Weiss model to a model in the sense of Definition 2.3, without affecting the semantics of sentences.

Proposition 3.1 (Translating Weiss models to our models)**.**
Let $M = \langle W, \leq, \mathcal{A}, \{R_X \mid X \in \mathcal{A}\}, V\rangle$ be a model in the sense of Weiss (2019b) (i.e., a structure defined as in Definition 2.3, but without the requirement that $R_X[w]$ is an up-set). Let $M^\sharp := \langle W, \leq, \mathcal{A}, \{R^\uparrow_X \mid X \subseteq W\}, V\rangle$, where:

$$R^\uparrow_X[w] = (R_X[w])^\uparrow := \{v \in W \mid \exists u \in R_X[w] \text{ such that } u \leq v\}$$

Then $M^\sharp$ is a model in the sense of Definition 2.3. Moreover, for any $w \in W$ and $\varphi \in \mathcal{L}^>$ we have:

$$M, w \models \varphi \iff M^\sharp, w \models \varphi$$

Proof. We leave it to the reader to verify that $M^\sharp$ satisfies all the conditions of Definition 2.3. The semantic preservation result is proved by induction on φ. The only interesting case is the inductive step for $\varphi = \psi > \chi$. We have:

$$\begin{aligned} M, w \models \varphi > \psi \quad &\iff \quad \forall v \in R_\varphi[w] : M, v \models \psi \\ &\iff \quad \forall v \in R_\varphi[w] : M^\sharp, v \models \psi \\ &\iff \quad \forall v \in R^\uparrow_\varphi[w] : M^\sharp, v \models \psi \\ &\iff \quad M^\sharp, w \models \varphi > \psi \end{aligned}$$

where the second equivalence is given by the induction hypothesis, and the third by the persistency of the semantics (Proposition 2.5). □

The main reason for working with a more demanding notion of models than used by Weiss is that this notion will allow for a smooth correspondence between semantic conditions on R_X and syntactic axioms on $>$. It is quite possible that similar results could be obtained for Weiss' original semantics as well, but it seems that the relevant conditions would have to be formulated in a more complex and less transparent way.[4]

[3]There is an additional difference between our semantics and the one in Weiss (2019a). Namely, in Weiss (2019a), it is required that a model provide relations R_X for every set of possible worlds $X \subseteq W$. This requirement, however, creates some unpleasant complications when constructing canonical models. Therefore, following Segerberg (1989), we take our models to come with a designated sub-algebra $\mathcal{A} \subseteq \wp(W)$ of propositions, and take our relations R_X to be indexed by elements $X \in \mathcal{A}$. Weiss' own more recent work (Weiss, 2019b), which we take as our point of departure here, makes the same assumption.

[4]In fact, the canonical model construction described in Weiss (2019b)—essentially the same we will use in this paper—delivers models where $R_X[w]$ is always upwards closed. Therefore, the upwards closure requirement does not make it more difficult to prove completeness for intuitionistic conditional logics. As we will discuss, the role of the requirement is, rather, to guarantee the soundness of certain axioms for certain natural frame conditions.

4 Intuitionistic conditional logics

In this section we introduce a class of *intuitionistic conditional logics*, defined proof-theoretically. In the next section we will see that ICK, the least logic in this class, is exactly the logic of the class of all intuitionistic conditional models—a result essentially due to Weiss (2019a). In the following sections we will then be concerned with stronger intuitionistic conditional logics, including the intuitionistic counterparts of Lewis's and Stalnaker's conditional logics.

Definition 4.1. An intuitionistic conditional logic (abbreviated as ICL) is a set $L \subseteq \mathcal{L}^{>}$ which includes all instances of the following schemata:

- Intuitionistic schemata:
  - $\varphi \to (\psi \to \varphi)$
  - $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
  - $\varphi \to (\psi \to \varphi \wedge \psi)$
  - $\varphi \wedge \psi \to \varphi$
  - $\varphi \wedge \psi \to \psi$
  - $\varphi \to \varphi \vee \psi$
  - $\psi \to \varphi \vee \psi$
  - $(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
  - $\bot \to \varphi$
- Conditional schemata
  - $\varphi > \top$
  - $(\varphi > \psi \wedge \chi) \leftrightarrow ((\varphi > \psi) \wedge (\varphi > \chi))$

and which is closed under the rules of *modus ponens*, *replacement of equivalent antecedents*, and *replacement of equivalent consequents*:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ (\mathsf{MP}) \qquad \frac{\varphi \leftrightarrow \psi}{(\varphi > \chi) \leftrightarrow (\psi > \chi)}\ (\mathsf{RCEA}) \qquad \frac{\varphi \leftrightarrow \psi}{(\chi > \varphi) \leftrightarrow (\chi > \psi)}\ (\mathsf{RCEC})$$

The minimal ICL, defined as the least set of formulas containing all instances of the above schemata and closed under the rules, is called ICK.

As in standard modal logic, some care must be taken in defining the relation of consequence $\vdash_L$ that the logic gives rise to. We define this as follows.

Definition 4.2 (Derivability in a logic)**.**
Let L be an ICL and let $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{>}$. We write $\Phi \vdash_L \psi$ to mean that there exist $\varphi_1, \ldots, \varphi_n \in \Phi$ such that $\varphi_1 \wedge \cdots \wedge \varphi_n \to \psi \in L$.

The following proposition states that in any ICL, conditionals are monotonic in the right component—a fact that will be useful in the following.

Proposition 4.3 (Right monotonicity)**.** (see Weiss, 2019a, Prop. 5)
Let L be an ICL and $\varphi, \psi, \chi \in \mathcal{L}^{>}$. If $\psi \vdash_L \chi$ then $\varphi > \psi \vdash_L \varphi > \chi$.

Proof. If $\psi \vdash_L \chi$ then $\psi \to \chi \in L$, and therefore by intuitionistic reasoning, $\psi \leftrightarrow \psi \wedge \chi \in L$. By closure under replacement of equivalent consequents, L contains the formula $(\varphi > \psi) \leftrightarrow (\varphi > \psi \wedge \chi)$. But L also contains the axiom $(\varphi > \psi \wedge \chi) \leftrightarrow (\varphi > \psi) \wedge (\varphi > \chi)$. By intuitionistic reasoning it follows that L contains the implication $(\varphi > \psi) \to (\varphi > \chi)$, which by definition means that $\varphi > \psi \vdash_L \varphi > \chi$. □

The notions of a logic L being sound and strongly complete with respect to a class of models $\mathcal{C}$ are defined in the standard way.

Definition 4.4 (Soundness, strong completeness)**.**
Let L be an ICL and $\mathcal{C}$ be a class of ICMs. We say that:

- L is *sound* for $\mathcal{C}$ if $\Phi \vdash_L \psi$ implies $\Phi \models_{\mathcal{C}} \psi$;
- L is *strongly complete* for $\mathcal{C}$ if $\Phi \models_{\mathcal{C}} \psi$ implies $\Phi \vdash_L \psi$.

5 Canonical model construction

In this section we describe how to get, for any intuitionistic conditional logic L, a canonical model M_L^c where non-entailment $\Phi \nvdash_L \psi$ is falsified at some world. This construction will be crucial to establishing completeness for the intuitionistic conditional logics that we will study below. As usual in intuitionistic logic, the model is based on consistent theories with the disjunction property.

Definition 5.1. Let $\Gamma \subseteq \mathcal{L}^{>}$ and let L be an ICL. We say that:

- Γ is an *L-theory* if for all $\varphi \in \mathcal{L}^{>}$: $\Gamma \vdash_L \varphi$ implies $\varphi \in \Gamma$;
- Γ is a *consistent* L-theory if Γ is an L-theory and $\bot \notin \Gamma$;
- Γ has the *disjunction property* if for all $\varphi, \psi \in \mathcal{L}^{>}$: $\varphi \vee \psi \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$.

The following lemma, familiar from intuitionistic logic, says that there are enough consistent L-theories with the disjunction property to witness all the non-entailments in L. The proof of the lemma is completely standard and therefore omitted (see, e.g., Lemma 11 in Bezhanishvili and de Jongh, 2006).

Lemma 5.2. If $\Phi \nvdash_L \psi$ then there exists a consistent L-theory with the disjunction property Γ such that $\Phi \subseteq \Gamma$ and $\psi \notin \Gamma$.

In the canonical model, a theory Γ' will be considered a refinement of another theory Γ if $\Gamma' \supseteq \Gamma$; moreover, Γ' will be considered a R^c_φ-successor of Γ if Γ' contains all formulas that are ontic consequences of φ according to Γ, that is, in case $\mathrm{Cn}_\varphi(\Gamma) \subseteq \Gamma'$, where the set $\mathrm{Cn}_\varphi(\Gamma)$ is defined as follows:

- $\mathrm{Cn}_\varphi(\Gamma) := \{\psi \in \mathcal{L}^> \mid \varphi > \psi \in \Gamma\}$

The following feature of $\mathrm{Cn}_\varphi(\Gamma)$ will play an important role below.

Lemma 5.3. If Γ is an L-theory, then so is $\mathrm{Cn}_\varphi(\Gamma)$.

Proof. Let Γ be an L-theory and suppose $\mathrm{Cn}_\varphi(\Gamma) \vdash_L \chi$. Then there are formulas $\psi_1, \ldots, \psi_n \in \mathrm{Cn}_\varphi(\Gamma)$ such that $\psi_1 \wedge \cdots \wedge \psi_n \vdash_L \chi$. By right monotonicity (Proposition 4.3) it follows that $(\varphi > \psi_1 \wedge \cdots \wedge \psi_n) \vdash_L (\varphi > \chi)$.

Since $\psi_1, \ldots, \psi_n \in \mathrm{Cn}_\varphi(\Gamma)$ we have $\varphi > \psi_1, \ldots, \varphi > \psi_n \in \Gamma$, and so also $(\varphi > \psi_1) \wedge \cdots \wedge (\varphi > \psi_n) \in \Gamma$. Since any conditional logic contains the axiom $(\varphi > \psi_1) \wedge \cdots \wedge (\varphi > \psi_n) \leftrightarrow (\varphi > (\psi_1 \wedge \cdots \wedge \psi_n))$, it follows that $\varphi > (\psi_1 \wedge \cdots \wedge \psi_n) \in \Gamma$. Since $\varphi > (\psi_1 \wedge \cdots \wedge \psi_n) \vdash_L \varphi > \chi$ and Γ is an L-theory, also $\varphi > \chi \in \Gamma$. Therefore, $\chi \in \mathrm{Cn}_\varphi(\Gamma)$. □

As usual, the canonical valuation function makes p satisfied at Γ in case $p \in \Gamma$. Summing up, then, the canonical model is defined as follows.

Definition 5.4 (Canonical model)**.**
The canonical model for an intuitionistic conditional logic L is the structure $M_L^c = \langle W_L^c, \leq^c, \mathcal{A}^c, \{R_\varphi^c \mid \varphi \in \mathcal{L}^>\}, V^c\rangle$, where:[5]

- W_L^c is the set of consistent L-theories with the disjunction property;
- $\Gamma \leq^c \Gamma' \iff \Gamma \subseteq \Gamma'$
- $\mathcal{A}^c = \{\widehat{\varphi} \mid \varphi \in \mathcal{L}^>\}$ where $\widehat{\varphi} := \{\Gamma \in W_L^c \mid \varphi \in \Gamma\}$
- $\Gamma R_{\widehat{\varphi}}^c \Gamma' \iff \mathrm{Cn}_\varphi(\Gamma) \subseteq \Gamma'$[6]
- $V^c(p) = \widehat{p}$

We need to make sure that $R_{\widehat{\varphi}}^c$ is well-defined, i.e., that if $\widehat{\varphi} = \widehat{\chi}$ then $R_{\widehat{\varphi}}^c = R_{\widehat{\chi}}^c$. This is guaranteed by the following proposition.

Proposition 5.5. If $\widehat{\varphi} = \widehat{\chi}$, then for all $\Gamma \in W_L^c$: $\mathrm{Cn}_\varphi(\Gamma) = \mathrm{Cn}_\chi(\Gamma)$.

Proof. Suppose $\widehat{\varphi} = \widehat{\chi}$. First, we claim that this implies $\varphi \leftrightarrow \chi \in L$. For suppose not: then either $\varphi \nvdash_L \chi$ or $\chi \nvdash_L \varphi$. Without loss of generality, suppose the former. By Lemma 5.2 there exists a theory Γ with $\varphi \in \Gamma$ and $\chi \notin \Gamma$, which means that $\Gamma \in \widehat{\varphi}$ but $\Gamma \notin \widehat{\chi}$, contrary to $\widehat{\varphi} = \widehat{\chi}$.

So, $\varphi \leftrightarrow \chi \in L$. By replacement of equivalent antecedents, it follows that for any ψ, $(\varphi > \psi) \leftrightarrow (\chi > \psi) \in L$. Now consider any $\Gamma \in W_L^c$. Since Γ is an L-theory, for any ψ we have $\varphi > \psi \in \Gamma \iff \chi > \psi \in \Gamma$. Since this holds for all ψ, it follows that $\mathrm{Cn}_\varphi(\Gamma) = \mathrm{Cn}_\chi(\Gamma)$. □

[5] Although all components of the model depend on L, in the case of the accessibility relations and the valuation function we omit the subscript L to improve readability.

[6] For this definition, it is crucial that we need to specify R_X^c only for $X \in \mathcal{A}^c$. If we had followed Weiss (2019a) in assuming that a model includes relations R_X for every $X \subseteq W_L^c$, we would have to define R_X^c even when X is not of the form $\widehat{\varphi}$. It is not clear how to do that in a principled way.

Moreover, the following proposition ensures that we have constructed an object of the right kind.

Proposition 5.6. M_L^c is an intuitionistic conditional model.

Proof. Clearly, the relation $\subseteq$ is a partial order on W_L^c. Every element of $\mathcal{A}^c$ is upwards closed, and $V(p) = \widehat{p} \in \mathcal{A}^c$. Three conditions remains to be shown:

- Closure of $\mathcal{A}^c$ under logic. We show only one case, namely, closure of $\mathcal{A}^c$ under the operation corresponding to $>$. Suppose $\widehat{\varphi}, \widehat{\chi} \in \mathcal{A}^c$.

  We claim that $\{\Gamma \mid R^c_{\widehat{\varphi}}[\Gamma] \subseteq \widehat{\chi}\} = \widehat{\varphi > \chi} \in \mathcal{A}^c$. This amounts to the claim that, for $\Gamma \in W_L^c$: $R^c_{\widehat{\varphi}}[\Gamma] \subseteq \widehat{\chi} \iff \varphi > \chi \in \Gamma$.

  Suppose $\varphi > \chi \in \Gamma$. Then $\chi \in \mathrm{Cn}_\varphi(\Gamma)$, therefore any $R^c_{\widehat{\varphi}}$-successor of Γ must contain χ, which means that $R^c_{\widehat{\varphi}}[\Gamma] \subseteq \widehat{\chi}$.

  Conversely, suppose $\varphi > \chi \notin \Gamma$. Then $\chi \notin \mathrm{Cn}_\varphi(\Gamma)$. By Lemma 5.3, $\mathrm{Cn}_\varphi(\Gamma) \nvdash_L \chi$. Therefore, by Lemma 5.2 there is $\Gamma' \in W_L^c$ with $\mathrm{Cn}_\varphi(\Gamma) \subseteq \Gamma'$ and $\chi \notin \Gamma'$. Thus, $\Gamma' \in R^c_{\widehat{\varphi}}[\Gamma]$ but $\Gamma' \notin \widehat{\chi}$, witnessing that $R^c_{\widehat{\varphi}}[\Gamma] \nsubseteq \widehat{\chi}$.

- $R^c_{\widehat{\varphi}}[\Gamma]$ is upwards closed. This is clear since, if $\Gamma' \subseteq \Gamma''$:

  $$\Gamma' \in R^c_{\widehat{\varphi}}[\Gamma] \iff \mathrm{Cn}_\varphi(\Gamma) \subseteq \Gamma' \Longrightarrow \mathrm{Cn}_\varphi(\Gamma) \subseteq \Gamma'' \iff \Gamma'' \in R^c_{\widehat{\varphi}}[\Gamma]$$

- $R^c_{\widehat{\varphi}}[\,\cdot\,]$ is monotonic. This is also clear: if $\Gamma \subseteq \Gamma'$ then $\mathrm{Cn}_\varphi(\Gamma) \subseteq \mathrm{Cn}_\varphi(\Gamma')$, and therefore:

  $$\Gamma'' \in R^c_{\widehat{\varphi}}[\Gamma'] \iff \mathrm{Cn}_\varphi(\Gamma') \subseteq \Gamma'' \Rightarrow \mathrm{Cn}_\varphi(\Gamma) \subseteq \Gamma'' \iff \Gamma'' \in R^c_{\widehat{\varphi}}[\Gamma]$$

□

Finally, we can prove that M_L^c behaves like a canonical model: satisfaction in a theory Γ amounts to membership in Γ.

Lemma 5.7 (Truth Lemma)**.**
For any $\Gamma \in W_L^c$ and any $\varphi \in \mathcal{L}^>$: $M_L^c, \Gamma \models \varphi \iff \varphi \in \Gamma$.

Proof. The proof is by induction on φ. We only give the inductive step for $\varphi = \chi > \psi$, since the other steps are exactly the same as in the completeness proof for intuitionistic logic. Recall that in the first item of the proof of Proposition 5.6 we proved that

$$\{\Gamma \mid R^c_{\widehat{\chi}}[\Gamma] \subseteq \widehat{\psi}\} = \widehat{\chi > \psi}$$

By induction hypothesis, $\widehat{\chi} = |\chi|$ and $\widehat{\psi} = |\psi|$. Using these facts, we have:

$$\begin{aligned} M_L^c, \Gamma \models \chi > \psi \quad &\iff \quad R^c_{\chi}[\Gamma] \subseteq |\psi| \\ &\iff \quad R^c_{\widehat{\chi}}[\Gamma] \subseteq \widehat{\psi} \\ &\iff \quad \Gamma \in \widehat{\chi > \psi} \\ &\iff \quad (\chi > \psi) \in \Gamma \end{aligned}$$

□

Notice that the truth-lemma can also be stated as follows: for all $\varphi \in \mathcal{L}^>$, in M_L^c we have $|\varphi| = \widehat{\varphi}$. Therefore, we also have $R_\varphi^c = R_{\widehat{\varphi}}^c$, which means that we can safely use the notation R_φ^c instead of the more cumbersome $R_{\widehat{\varphi}}^c$.

Using the Truth-Lemma we can show that any entailment which is not valid in L can be falsified in the canonical model for L.

Proposition 5.8. If $\Phi \not\vdash_L \psi$, then there exists a world $\Gamma \in W_L^c$ such that $M_L^c, \Gamma \models \varphi$ for all $\varphi \in \Phi$ but $M_L^c, \Gamma \not\models \psi$.

Proof. Suppose $\Phi \not\vdash_L \psi$. By Lemma 5.2 there exists $\Gamma \in W_L^c$ such that $\Phi \subseteq \Gamma$ but $\psi \notin \Gamma$. By the truth-lemma, in the canonical model Γ satisfies all formulas in Φ and does not satisfy ψ. □

6 Semantic constraints

In this section we introduce the intuitionistic counterpart of several well-known constraints on the accessibility relations R_φ, and the impact that these constraints have on the resulting conditional logic. The constraints that we will consider are listed in the following definition, where in each case the variables φ, ψ and w are understood to be universally quantified.[7,8]

Definition 6.1 (Semantic conditions)**.**

C1. $R_\varphi[w] \subseteq |\varphi|$

C2. if $w \in |\varphi|$ then $w \in R_\varphi[w]$[9]

C3. if $R_\varphi[w] = \emptyset$ then $R_{\varphi\wedge\psi}[w] = \emptyset$

C4. if $R_\varphi[w] \cap |\psi| \neq \emptyset$ then $R_{\varphi\wedge\psi}[w] \subseteq R_\varphi[w]$

[7] We formulate the relevant constraints for relations of the form R_φ, i.e., relations R_X where $X = |\varphi|$ for some $\varphi \in \mathcal{L}^>$. It would be more natural to formulate these constraints for all R_X, regardless of whether X is definable by a formula or not. However, we prefer to use formulas as indices, since then the significance of the constraints becomes easier to grasp. Which of the two formulations we choose does not make a big difference, since relations R_X where X is not definable are immaterial to the semantics. Thus, e.g., even though requiring $R_\varphi[w] \subseteq |\varphi|$ for all $\varphi \in \mathcal{L}^>$ is strictly weaker than requiring $R_X[w] \subseteq X$ for all $X \in \mathcal{A}$, the two requirements lead to classes of models which validate exactly the same logic.

[8] These conditions are natural generalizations to the intuitionistic setting of conditions which are standard in the field of conditional logics. In the classical setting, conditions C1—C5 would look exactly the same, while conditions C6 and C7 would be stated by replacing $w^\uparrow$ and $v^\uparrow$ by $\{w\}$ and $\{v\}$ (as would result from taking $\leq$ to be the identity relation). These conditions go back to the work of Lewis (1973), where they are stated as constraints on a selection function f mapping a world w and a sentence φ to a set $f(\varphi, w) \subseteq W$. For a presentation in terms of relations, see Segerberg (1989). Both Lewis and Segerberg do not exactly divide labor between the different constraints in the way we do. Here we have chosen the division of labor that seemed more natural to us, both to isolate the conceptual significance of each constraint, and to obtain a smooth correspondence with axiom schemata.

[9] The reader should not confuse condition C2 with Stalnaker's conditional logic, `C2`. Throughout the paper we use sans-serif fonts for semantic conditions and the corresponding axioms, and we use typewriter fonts for specific conditional logics.

C5. $R_\varphi[w] \cap |\psi| \subseteq R_{\varphi\wedge\psi}[w]$

C6. if $w \in |\varphi|$ then $R_\varphi[w] \subseteq w^\uparrow$

C7. if $R_\varphi[w] \neq \emptyset$ then $R_\varphi[w] = v^\uparrow$ for some v

Let us briefly discuss the conceptual significance and the logical repercussions of each condition.

Condition C1 is a success constraint: it says that making the assumption that φ should lead to a hypothetical context $R_\varphi[w]$ containing only φ-worlds. In other words, to judge a conditional $\varphi > \psi$ is to judge whether ψ holds in certain φ-worlds. This condition ensures that $\varphi > \psi$ is implied by the universal strict conditional, in the following sense.

Observation 6.2.
Suppose C1 holds in M. If $|\varphi| \subseteq |\psi|$, then $M, w \models \varphi > \psi$ for all $w \in W$.

Condition C2 is known as the *weak centering* condition: it says that if φ is actually true w, then w is one of the worlds which might have been the case if φ had been the case. That is, if the antecedent is true, then the actual world is relevant to determining the truth of the conditional. In our intuitionistic setting, C2 implies another interesting property: in every world w, any refinement of w which satisfies φ is relevant to the truth of a conditional $\varphi > \psi$. To state this precisely, recall that we used the notation $S_\varphi[w]$ for the set of refinements of w which satisfy φ, that is, $S_\varphi[w] = \{v \geq w \mid M, v \models \varphi\}$. We have the following.

Proposition 6.3.
Suppose C2 holds in M. Then for any w and any φ: $S_\varphi[w] \subseteq R_\varphi[w]$.

Proof. Suppose $v \in S_\varphi[w]$. This means that $v \geq w$ and $v \in |\varphi|$. By C2, $v \in R_\varphi[v]$. By the monotonicity of R_φ, $R_\varphi[v] \subseteq R_\varphi[w]$. Therefore, $v \in R_\varphi[w]$. ☐

Since $S_\varphi[w]$ and $R_\varphi[w]$ provide, respectively, the domains of quantifications used to assess $\varphi \to \psi$ and $\varphi > \psi$, this proposition implies that, given C2, $\varphi > \psi$ is at least as strong as the intuitionistic conditional.

Corollary 6.4.
Suppose C2 holds in M. Then $M, w \models \varphi > \psi$ implies $M, w \models \varphi \to \psi$.

Thus, conditions C1 and C2 together imply that the conditional $>$ in intermediate in strength between the universal strict conditional and the intuitionistic conditional. So, the two interpretations of $>$ considered in Definition 2.6 are, respectively, the weakest and the strongest interpretation of $>$ compatible with C1 and C2.

Condition C3 simply says that if φ cannot be consistently supposed, then anything that implies φ cannot be consistently supposed either.

Condition C4 is a cautious monotonicity constraint. It says that, when we strengthen an antecedent from φ to $\varphi \wedge \psi$, we must get a stronger hypothetical context $R_{\varphi\wedge\psi}[w] \subseteq R_\varphi[w]$, as long as the stronger antecedent is still compatible with the hypothetical context determined by the weaker antecedent.

Condition $\mathsf{C5}$ is also about the effect of strengthening an antecedent. It says that if v is one of the ways in which things might be if φ were the case, and if v also satisfies ψ, then v is one of the ways in which things might be if $\varphi \wedge \psi$ were the case.

Notice that $\mathsf{C1}$, $\mathsf{C4}$, and $\mathsf{C5}$ together completely determine the effect of strengthening an antecedent in those cases where the stronger antecedent is consistent with the hypothetical context for the weak one.

Proposition 6.5.
Let M obey $\mathsf{C1}$, $\mathsf{C4}$, and $\mathsf{C5}$. If $R_\varphi[w] \cap |\psi| \neq \emptyset$, then $R_{\varphi\wedge\psi}[w] = R_\varphi[w] \cap |\psi|$.

Proof. $\mathsf{C1}$ gives the inclusion $R_{\varphi\wedge\psi}[w] \subseteq |\psi|$, and $\mathsf{C4}$ the inclusion $R_{\varphi\wedge\psi}[w] \subseteq R_\varphi[w]$. Thus, together these conditions imply $R_{\varphi\wedge\psi}[w] \subseteq R_\varphi[w] \cap |\psi|$. The converse inclusion is given by $\mathsf{C5}$. □

Condition $\mathsf{C6}$ says that, if φ is true at w, then no *counterfactual* world—i.e., no world which is not a refinement of w—is relevant to determining the truth of $\varphi > \psi$ at w. In combination with $\mathsf{C2}$, this gives the *strong centering* condition, which in our setting is formulated as follows.

Proposition 6.6.
Suppose $\mathsf{C2}$ and $\mathsf{C6}$ hold in M. If $w \in |\varphi|$, then $R_\varphi[w] = w^\uparrow$.

This condition looks a bit different than the classical strong centering condition, which requires that, if $w \in |\varphi|$ then $R_\varphi[w] = \{w\}$. However, note that the classical formulation of strong centering would not be compatible with the upwards-closure requirement on ICMs, since the set $\{w\}$ is not upwards-closed if w is not an endpoint (i.e., if there are proper extensions $v > w$). If $R_\varphi[w]$ includes w, then by upwards closure it must contain all the set $w^\uparrow$. Thus, $R_\varphi[w] = w^\uparrow$ is the smallest hypothetical context which includes w.

Conceptually, the point can be put as follows. Suppose φ is actually true. Then the worlds that might be the case if φ were the case are just those worlds which might *in fact* be the case, and these are exactly the refinements of the actual state of affairs.

Note that if we look at the special case of classical conditional models, i.e., models where where the relation $\leq$ is the identity (see the discussion under Observation 2.7), then we retrieve the standard formulation of strong centering, since in that case $w^\uparrow = \{w\}$.

Finally, and most importantly, in our setting the condition $R_\varphi[w] = w^\uparrow$ captures exactly the idea of strong centering: if φ is true at w, then the only world which is relevant to assessing the truth of a conditional $\varphi > \psi$ is w itself. This is brought out most clearly by the following proposition.

Proposition 6.7. Suppose $\mathsf{C2}$ and $\mathsf{C6}$ hold in M. Then if $w \in |\varphi|$, for every ψ we have $M, w \models \varphi > \psi \iff M, w \models \psi$.

Proof. If $\mathsf{C2}$ and $\mathsf{C6}$ hold and $w \in |\varphi|$, then by the previous proposition we have $R_\varphi[w] = w^\uparrow$. Now suppose $M, w \models \varphi > \psi$. Then $R_\varphi[w] \subseteq |\psi|$, and since

$w \in w^{\uparrow} = R_{\varphi}[w]$ it follows that $M, w \models \psi$. Conversely, suppose $M, w \models \psi$. Then by persistency (Proposition 2.5), every $v \geq w$ satisfies ψ as well, so we have $w^{\uparrow} \subseteq |\psi|$. Since $R_{\varphi}[w] = w^{\uparrow}$, it follows that $M, w \models \varphi > \psi$. □

Finally, C7 is the intuitionistic counterpart of Stalnaker's uniqueness assumption. Again, in order to obey upwards-closure, the formulation of the assumption is slightly different than in the classical case, where $R_{\varphi}[w] = v^{\uparrow}$ would be replaced by $R_{\varphi}[w] = \{v\}$. However, again the classical formulation is retrieved when we restrict to classical models, where $\leq$ is the identity. Moreover, the fundamental idea of Stalnaker's assumption is precisely retained: in order to assess conditionals with antecedent φ we just need to consider what is the case in a *single* possible world. This is formalized by the following proposition.

Proposition 6.8. Suppose C7 holds in M. If $R_{\varphi}[w] \neq \emptyset$, then there exists a world v such that for every $\psi \in \mathcal{L}^{>}$: $M, w \models \varphi > \psi \iff M, v \models \psi$.

Proof. Suppose $R_{\varphi}[w] \neq \emptyset$. By C7 there exists a world v such that $R_{\varphi}[w] = v^{\uparrow}$. Then for every $\psi \in \mathcal{L}^{>}$ we have: $M, w \models \varphi > \psi \iff R_{\varphi}[w] \subseteq |\psi| \iff v^{\uparrow} \subseteq |\psi| \iff M, v \models \psi$, where the last bi-conditional uses the persistency property of the semantics. □

In the classical setting, weak centering and the uniqueness assumption jointly imply strong centering. Suppose $w \in |\varphi|$, by weak centering $w \in R_{\varphi}[w]$, and by the uniqueness assumption $R_{\varphi}[w]$ is a singleton. Therefore we must have $R_{\varphi}[w] = \{w\}$. The same is not true in the intuitionistic setting. Suppose $w \in |\varphi|$, then by weak centering $w \in R_{\varphi}[w]$, and by the uniqueness assumption $R_{\varphi}[w] = v^{\uparrow}$ for some v. And yet this is perfectly compatible with w being a proper refinement of v, contrary to strong centering. Thus, strong centering and the uniqueness assumptions are, even given weak centering, independent requirements in the intuitionistic setting.

Finally, it is interesting to remark which of the conditions above are satisfied by the models M^u and M^i of Definition 2.6 which yield, respectively, the universal strict conditional and the intuitionistic conditional.

Observation 6.9. Let M be an intuitionistic Kripke model. Then:

- M^u satisfies conditions C1–C5 but, in general, not C6 and C7.[10]
- M^i satisfies conditions C1–C6 but, in general, not C7.[11]

[10]To see that M^u does not generally satisfy C6 and C7, consider an intuitionistic Kripke model M with three points w, v, u, where w is the root and v, u are endpoints. Suppose $V(p) = \{v, u\}$. Then in M^u we have $R_p[w] = |p| = \{v, u\}$, which is not a rooted set, violating C7. Moreover we have $R_{\top}[v] = |\top| = W \not\subseteq v^{\uparrow}$, violating C6.

[11]To see that M^i does not generally satisfy C7, take the model M of the previous footnote. In M^i we have $R_p[w] = |p| \cap w^{\uparrow} = \{v, u\}$, which is not a rooted set, violating C7.

7 Conditional axioms

In the previous section, we considered several semantic constraints on the conditional accessibility relations R_φ. In this section we will introduce corresponding axioms for the conditional operator $>$, and show that each of them is valid on the class of models where the corresponding condition holds. In the next section we will extend this to a general soundness and completeness result for logics obtained by expanding ICK with one or more of our axioms.[12]

Definition 7.1 (Conditional axiom schemata)**.**
We will be concerned with the following schemata, where φ, ψ, χ range over $\mathcal{L}^{>}$.

A1. $\varphi > \varphi$

A2. $(\varphi > \psi) \to (\varphi \to \psi)$

A3. $(\varphi > \bot) \to (\varphi \land \psi > \bot)$

A4. $(\varphi > \chi) \to ((\varphi > \neg\psi) \lor ((\varphi \land \psi) > \chi))$

A5. $(\varphi \land \psi > \chi) \to (\varphi > (\psi \to \chi))$

A6. $\varphi \land \psi \to (\varphi > \psi)$

A7. $(\varphi > \psi \lor \chi) \to (\varphi > \psi) \lor (\varphi > \chi)$

In the next proposition we show that each of these axioms is valid on models satisfying the corresponding semantic condition.

Theorem 7.2 (Soundness)**.** For $1 \leq \mathsf{i} \leq 7$, every instance of the schema Ai is valid with respect to the class of models satisfying condition Ci.

Proof. 1. Suppose M satisfies $\mathsf{C1}$. Then for any world w we have $R_\varphi[w] \subseteq |\varphi|$, which means that $M, w \models \varphi > \varphi$. So Axiom 1 is valid.

2. Suppose M satisfies $\mathsf{C2}$. To show that the implication $(\varphi > \psi) \to (\varphi \to \psi)$ is valid in the model we just need to show that in all worlds where the antecedent is true, the consequent is true. So, take a world w and suppose $M, w \models \varphi > \psi$. We want to show that $M, w \models \varphi \to \psi$. Take a successor $v \geq w$ and suppose $M, v \models \varphi$. By persistency, also $M, v \models \varphi > \psi$, which means that $R_\varphi[v] \subseteq |\psi|$. By $\mathsf{C2}$, $v \in R_\varphi[v]$. Therefore, $M, v \models \psi$. This shows that $M, w \models \varphi \to \psi$, as we wanted.

3. Suppose M satisfies $\mathsf{C3}$. Suppose that $M, w \models \varphi > \bot$. Then $R_\varphi[w] = \emptyset$. By $\mathsf{C3}$ we have $R_{\varphi\land\psi}[w] = \emptyset$, and therefore $M, w \models \varphi \land \psi > \bot$.

[12] These axioms are standard in the area of conditional logic (see, e.g., Stalnaker and Thomason, 1970; Lewis, 1973; Nute, 1980; Segerberg, 1989; Arlo-Costa *et al.*, 2019); however, as we will discuss below, in our setting one must choose carefully among classically equivalent formulations of the relevant principles. The formulations given here are chosen in such a way as to correspond to the frame conditions in Definition 6.1.

4. Suppose M satisfies C4. Suppose that $M, w \models \varphi > \chi$, i.e., $R_\varphi[w] \subseteq |\chi|$. We want to show that $M, w \models (\varphi \to \neg\psi) \vee (\varphi \wedge \psi > \chi)$. We distinguish two cases:

   - Case 1: $R_\varphi[w] \cap |\psi| = \emptyset$. Take any $v \in R_\varphi[w]$. Since $R_\varphi[w]$ is upwards closed, for any successor $u \geq v$ we have $u \in R_\varphi[w]$, and therefore $u \notin |\psi|$. This means that $M, v \models \neg\psi$. Therefore, $M, w \models \varphi > \neg\psi$.
   - Case 2: $R_\varphi[w] \cap |\psi| \neq \emptyset$. In this case, C4 implies $R_{\varphi\wedge\psi}[w] \subseteq R_\varphi[w]$, and since $R_\varphi[w] \subseteq |\chi|$ we have $M, w \models \varphi \wedge \psi > \chi$.

   In both cases, $M, w \models (\varphi \to \neg\psi) \vee (\varphi \wedge \psi > \chi)$.

5. Suppose M satisfies C5. Take a world w with $M, w \models \varphi \wedge \psi > \chi$. Then $R_{\varphi\wedge\psi}[w] \subseteq |\chi|$. Take any $v \in R_\varphi[w]$. We want to show $v \models \psi \to \chi$.

   So, consider any $u \geq v$ with $M, u \models \psi$. Since $R_\varphi[w]$ is upwards closed, $u \in R_\varphi[w]$; and since $M, u \models \psi$, $u \in |\psi|$. Thus, $u \in R_\varphi[w] \cap |\psi| \subseteq R_{\varphi\wedge\psi}[w] \subseteq |\chi|$, where the first inclusion is given by C5. Hence, $M, u \models \chi$.

   This shows that $M, v \models \psi \to \chi$. Since v was any world in $R_\varphi[w]$, it follows that $M, w \models \varphi > (\psi \to \chi)$.

6. Suppose M satisfies C6. Take a world w with $M, w \models \varphi \wedge \psi$. Since $w \in |\varphi|$, by C6 we have $R_\varphi[w] \subseteq w^\uparrow$. Since $w \in |\psi|$, by the persistency of the semantics we have $w^\uparrow \subseteq |\psi|$. Therefore, $R_\varphi[w] \subseteq |\psi|$, which means that $M, w \models \varphi > \psi$.

7. Suppose M satisfies C7. Take a world w with $M, w \models \varphi > \psi \vee \chi$. This means that $R_\varphi[w] \subseteq |\psi \vee \chi|$. We want to show $M, w \models (\varphi > \psi) \vee (\varphi > \chi)$.

   If $R_\varphi[w] = \emptyset$ the conclusion follows trivially, so we may assume $R_\varphi[w] \neq \emptyset$. Then, by C7 there is a world v such that $R_\varphi[w] = v^\uparrow$. Since $v \in R_\varphi[w] \subseteq |\psi \vee \chi|$, $M, v \models \psi \vee \chi$, so either $M, v \models \psi$, or $M, v \models \chi$.

   Suppose the former. Then by persistency, every successor of v validates ψ as well, which means that $R_\varphi[w] = v^\uparrow \subseteq |\psi|$, which implies $M, w \models \varphi > \psi$.

   Reasoning analogously, if $M, v \models \chi$ we conclude $M, w \models \varphi > \chi$. In either case, it follows that $M, w \models (\varphi > \psi) \vee (\varphi > \chi)$.

□

Notice that, in order to show that all instances A4 and A5 are valid on the corresponding class of models, we made crucial use of the upwards closure condition of $R_\varphi[w]$, which is required by our notion of models, but not by Weiss (2019a,b). It is not hard to show that, if $R_\varphi[w]$ is not required to be upwards closed, then not all instances of A4 and A5 are valid with respect to the classes of models defined by the corresponding conditions. If one does not wish to make the assumption that $R_\varphi[w]$ is upwards closed, one may look for ways to strengthen the semantic conditions C4 and C5 in such a way as to render the corresponding schemata valid. However, since the assumption that $R_\varphi[w]$ is upwards-closed seems natural in the intuitionistic setting, and does not make the semantics less general, as shown in Section 3, we prefer to make this assumption and keep the semantic conditions as simple as possible.

8 Canonicity and completeness

In this section, we prove a general completeness result for logics obtained by extending ICK with any combination of the above axiom schemata. The crucial part of the proof is to show that each schema is *canonical* for the corresponding semantic property: that is, if a logic L includes all instances of the schema, then the canonical model for L has the relevant property.

Proposition 8.1 (Canonicity)**.**
For $1 \leq i \leq 7$, if an intuitionistic conditional logic L includes all instances of Ai, then M^c_L satisfies condition Ci.

Proof.

1. Suppose L contains all instances of $\mathsf{A1}$. We want to show that, for all points $\Gamma \in W^c_L$ and formulas φ we have $R^c_\varphi[\Gamma] \subseteq |\varphi|$.

   Since $\varphi > \varphi \in L$ and Γ is an L-theory, $\varphi > \varphi \in \Gamma$, therefore $\varphi \in \mathrm{Cn}_\varphi(\Gamma)$. Now take any $\Gamma' \in R^c_\varphi[\Gamma]$ we have $\varphi \in \mathrm{Cn}_\varphi(\Gamma) \subseteq \Gamma'$. By the Truth Lemma, this implies $M^c_L, \Gamma' \models \varphi$. This shows that $R^c_\varphi[\Gamma] \subseteq |\varphi|$.

2. Suppose L contains all instances of $\mathsf{A2}$. Suppose $\Gamma \in |\varphi|$, which by the Truth Lemma means that $\varphi \in \Gamma$. We want to show that $\Gamma \in R^c_\varphi[\Gamma]$.

   Consider any $\psi \in \mathrm{Cn}_\varphi(\Gamma)$. This means that $\varphi > \psi \in \Gamma$. Since Γ is an L-theory and L contains $(\varphi > \psi) \to (\varphi \to \psi)$, also $\varphi \to \psi \in \Gamma$. Since $\varphi \in \Gamma$, it follows $\psi \in \Gamma$.

   We have shown that $\mathrm{Cn}_\varphi(\Gamma) \subseteq \Gamma$, which by definition implies $\Gamma \in R^c_\varphi[\Gamma]$.

3. Suppose L contains all instances of $\mathsf{A3}$. Suppose $R^c_\varphi[\Gamma] = \emptyset$. We want to show that also $R^c_{\varphi\wedge\psi}[\Gamma] = \emptyset$.

   First, notice that $R^c_\varphi[\Gamma] = \emptyset$ implies that $\mathrm{Cn}_\varphi(\Gamma) \vdash_L \bot$. For otherwise, by Lemma 5.2, $\mathrm{Cn}_\varphi(\Gamma)$ could be extended to a world $\Gamma' \in W^c_L$, and then we would have $\Gamma' \in R^c_\varphi[\Gamma]$.

   By Lemma 5.3, $\mathrm{Cn}_\varphi(\Gamma) \vdash_L \bot$ implies $\bot \in \mathrm{Cn}_\varphi(\Gamma)$, that is, $\varphi > \bot \in \Gamma$. Since Γ is an L-theory and L contains $(\varphi > \bot) \to (\varphi \wedge \psi > \bot)$, we have $\varphi \wedge \psi > \bot \in \Gamma$, that is, $\bot \in \mathrm{Cn}_{\varphi\wedge\psi}(\Gamma)$. Now for every $\Gamma' \in W^c_L$ we have $\bot \notin \Gamma'$, and therefore $\mathrm{Cn}_{\varphi\wedge\psi}(\Gamma) \not\subseteq \Gamma'$. This shows that $R^c_{\varphi\wedge\psi}[\Gamma'] = \emptyset$.

4. Suppose L contains all instances of $\mathsf{A4}$. Suppose $R^c_\varphi[\Gamma] \cap |\psi| \neq \emptyset$. This means that there exists Γ' such that $\Gamma' \in R^c_\varphi[\Gamma]$ and $\Gamma' \in |\psi|$. By definition of the canonical accessibility relation R^c_φ, $\Gamma' \in R^c_\varphi[\Gamma]$ implies $\mathrm{Cn}_\varphi(\Gamma) \subseteq \Gamma'$. By the truth-lemma, $\Gamma' \in |\psi|$ implies $\psi \in \Gamma'$. Thus, $\mathrm{Cn}_\varphi(\Gamma) \cup \{\psi\} \subseteq \Gamma'$. This implies that $\neg\psi \notin \mathrm{Cn}_\varphi(\Gamma)$, since otherwise Γ' would include both ψ and $\neg\psi$, and could not be a consistent theory. Hence, $\varphi > \neg\psi \notin \Gamma$.

We want to show that $R^c_{\varphi\wedge\psi}[\Gamma] \subseteq R^c_\varphi[\Gamma]$. This will follow if we can show that $\mathrm{Cn}_{\varphi\wedge\psi}(\Gamma) \supseteq \mathrm{Cn}_\varphi(\Gamma)$, since then we have:

$$\Gamma' \in R^c_{\varphi\wedge\psi}[\Gamma] \iff \mathrm{Cn}_{\varphi\wedge\psi}(\Gamma) \subseteq \Gamma' \Longrightarrow \mathrm{Cn}_\varphi(\Gamma) \subseteq \Gamma' \iff \Gamma' \in R^c_\varphi[\Gamma]$$

So, take $\chi \in \mathrm{Cn}_\varphi(\Gamma)$. This means that $\varphi > \chi \in \Gamma$. Since Γ is an L-theory and L includes $(\varphi > \chi) \to ((\varphi > \neg\psi) \vee (\varphi \wedge \psi > \chi))$, it follows that $(\varphi > \neg\psi) \vee (\varphi \wedge \psi > \chi) \in \Gamma$. Since Γ has the disjunction property, one of the disjuncts is in Γ. Since we already know that $\varphi > \neg\psi \notin \Gamma$, it follows that $\varphi \wedge \psi > \chi \in \Gamma$. Therefore, $\chi \in \mathrm{Cn}_{\varphi\wedge\psi}(\Gamma)$, as we wanted.

5. Suppose L contains all instances of A5. We want to show that $R^c_\varphi[\Gamma] \cap |\psi| \subseteq R^c_{\varphi\wedge\psi}[\Gamma]$.

   Take any $\Gamma' \in R^c_\varphi[\Gamma] \cap |\psi|$: by the Truth Lemma, this means that $\mathrm{Cn}_\varphi(\Gamma) \subseteq \Gamma'$ and $\psi \in \Gamma'$. We need to prove that $\Gamma' \subseteq R^c_{\varphi\wedge\psi}[\Gamma]$, which amounts to showing that $\mathrm{Cn}_{\varphi\wedge\psi}(\Gamma) \subseteq \Gamma'$.

   Suppose $\chi \in \mathrm{Cn}_{\varphi\wedge\psi}(\Gamma)$. This means that $\varphi \wedge \psi > \chi \in \Gamma$. Since Γ is an L-theory and L contains $(\varphi \wedge \psi > \chi) \to (\varphi > (\psi \to \chi))$, it follows that $\varphi > (\psi \to \chi) \in \Gamma$. Thus, $\psi \to \chi \in \mathrm{Cn}_\varphi(\Gamma) \subseteq \Gamma'$. Finally, since $\psi \in \Gamma'$ and $\psi \to \chi \in \Gamma'$, it follows that $\chi \in \Gamma'$.

   Therefore, we have shown that $\mathrm{Cn}_{\varphi\wedge\psi}(\Gamma) \subseteq \Gamma'$, as we wanted.

6. Suppose L contains all instances of A6. Take a point $\Gamma \in |\varphi|$, which by the Truth Lemma means that $\varphi \in \Gamma$. We want to show that $R^c_\varphi[\Gamma] \subseteq \Gamma^\uparrow$.

   Take any $\Gamma' \in R^c_\varphi[\Gamma]$, which means that $\mathrm{Cn}_\varphi(\Gamma) \subseteq \Gamma'$. We need to show that $\Gamma' \in \Gamma^\uparrow$, which amounts to $\Gamma \subseteq \Gamma'$. This will follow if we can show that $\Gamma \subseteq \mathrm{Cn}_\varphi(\Gamma)$.

   Take any $\psi \in \Gamma$. Since $\varphi, \psi \in \Gamma$, also $\varphi \wedge \psi \in \Gamma$. Since Γ is an L-theory and L includes $\varphi \wedge \psi \to (\varphi > \psi)$, it follows that $\varphi > \psi \in \Gamma$, so $\psi \in \mathrm{Cn}_\varphi(\Gamma)$. Therefore, $\Gamma \subseteq \mathrm{Cn}_\varphi(\Gamma)$, as we wanted.

7. Suppose L contains all instances of A7. Suppose $R^c_\varphi[\Gamma] \neq \emptyset$. We need to show that $R^c_\varphi[\Gamma]$ is rooted, i.e., there exists a $\Gamma' \in W^c_L$ s.t. $R^c_\varphi[\Gamma] = (\Gamma')^\uparrow$.

   If we can show that $\mathrm{Cn}_\varphi(\Gamma) \in W^c_L$, then we are done, since then we have that $R^c_\varphi[\Gamma] = \{\Gamma' \in W^c_L \mid \mathrm{Cn}_\varphi(\Gamma) \subseteq \Gamma'\} = \mathrm{Cn}_\varphi(\Gamma)^\uparrow$.

   So, we need to show that $\mathrm{Cn}_\varphi(\Gamma)$ is a consistent L-theory with the disjunction property. By Lemma 5.3, $\mathrm{Cn}_\varphi(\Gamma)$ is an L-theory. Moreover, $\mathrm{Cn}_\varphi(\Gamma)$ is consistent, since if $\bot \in \mathrm{Cn}_\varphi(\Gamma)$ then no consistent theory could include $\mathrm{Cn}_\varphi(\Gamma)$, and therefore $R^c_\varphi[\Gamma] = \emptyset$, contrary to assumption.

   It remains to be shown that $\mathrm{Cn}_\varphi(\Gamma)$ has the disjunction property. Let $\psi \vee \chi \in \mathrm{Cn}_\varphi(\Gamma)$. This means that $(\varphi > \psi \vee \chi) \in \Gamma$. Since Γ is an L-theory and L includes $(\varphi > \psi \vee \chi) \to (\varphi > \psi) \vee (\varphi > \chi)$, it follows that $(\varphi > \psi) \vee (\varphi > \chi) \in \Gamma$. Since Γ has the disjunction property, it follows that $\varphi > \psi \in \Gamma$ or $\varphi > \chi \in \Gamma$. Therefore, $\psi \in \mathrm{Cn}_\varphi(\Gamma)$ or $\chi \in \mathrm{Cn}_\varphi(\Gamma)$. Thus, $\mathrm{Cn}_\varphi(\Gamma)$ has the disjunction property, which completes the proof.

□

Using this fact, we are now ready to show our main result: any subset of the schemata A1–A7 gives rise to a logic which is sound and complete with respect to the associated class of models. In order to state it precisely, let us introduce some notation.

Definition 8.2. Given any subset $\{i_1, \ldots, i_n\} \subseteq \{1, \ldots, 7\}$:

- $L(i_1, \ldots, i_n)$ is the least ICL containing all instances of schemata $\mathsf{A}i_1, \ldots, \mathsf{A}i_n$;
- $\mathcal{C}(i_1, \ldots, i_n)$ is the set of ICMs satisfying conditions $\mathsf{C}i_1, \ldots, \mathsf{C}i_n$.

Then our main result can be stated precisely as follows.

Theorem 8.3 (Soundness and completeness)**.**
Let $\{i_1, \ldots, i_n\} \subseteq \{1, \ldots, 7\}$. The logic $L(i_1, \ldots, i_n)$ is sound and strongly complete for the class $\mathcal{C}(i_1, \ldots, i_n)$.

Proof. The soundness direction follows from the fact that each axiom is valid on models having the corresponding property (Theorem 7.2). For the completeness direction, suppose $\Phi \not\vdash_{L(i_1,\ldots,i_n)} \psi$. By Prop. 5.8 there exists a world Γ in the canonical model $M^c_{L(i_1,\ldots,i_n)}$ such that $M^c_{L(i_1,\ldots,i_n)}, \Gamma \models \varphi$ for all $\varphi \in \Phi$, but $M^c_{L(i_1,\ldots,i_n)}, \Gamma \not\models \psi$. By Proposition 8.1, $M^c_{L(i_1,\ldots,i_n)} \in \mathcal{C}(i_1, \ldots, i_n)$. Therefore, $\Phi \not\models_{\mathcal{C}(i_1,\ldots,i_n)} \psi$. □

As a particular case of this general theorem, we get soundness and completeness theorems for some natural intuitionistic counterparts of the classical conditional logics `V`, `VW`, `VC`, and `C2`.[13]

Definition 8.4. We define the following logics:

- `IV` $:= L(1, 3, 4, 5)$
- `IVW` $:= L(1, 2, 3, 4, 5)$
- `IVC` $:= L(1, 2, 3, 4, 5, 6)$
- `IC2` $:= L(1, 2, 3, 4, 5, 6, 7)$

In the classical setting, axiom A6 is derivable from the remaining axioms, and therefore not strictly needed to axiomatize the logic `C2`. Interestingly, this is not the case in the intuitionistic setting.

Proposition 8.5. $\not\vdash_{L(1,2,3,4,5,7)} (p \wedge q) \to (p > q)$

[13]Notice that, from a formal point of view, a classical conditional logic L_c will, in general, have many intuitionistic counterparts; that is, there will be multiple intuitionistic conditional logics L_i with the property that L_i augmented with excluded middle is L_c. However, the logics `IV`, `IVW`, `IVC`, and `IC2` do not just correspond to `V`, `VW`, `VC`, and `C2` in this weak sense, but also in a stronger sense. Namely, as we will show momentarily, they arise from semantic conditions which are the natural intuitionistic generalizations of those conditions which, in the classical case, give rise to the logics `V`, `VW`, `VC`, and `C2`.

Proof. Consider an intuitionistic Kripke model M with two points w and v, where $w < v$. Suppose $V(p) = \{w, v\}$ and $V(q) = \{v\}$. Now consider the model M^u obtained by letting $R_X[w] := X$, as in Definition 2.6. By Observation 6.9, this model satisfies conditions C1–C5. Moreover, condition C7 is satisfied as well, since M^u is finite and linear. It follows from Proposition 7.2 that axioms A1–A5 and A7 are valid in this model. Yet, we have $M^u, w \not\models p \wedge q \to (p > q)$. To see this, notice that $M^u, v \models p \wedge q$, but $M^u, v \not\models p > q$, because $R_p[v] = |p| = \{w, v\}$ and $M^u, w \not\models q$. □

The completeness results for IV, IVW, IVC, and IC2 are special cases of Theorem 8.3.

Corollary 8.6 (Soundness and completeness for IVW, IVC, IC2)**.**

- IV is sound and complete for the class of models satisfying C1 and C3-C5.
- IVW is sound and complete for the class of models satisfying C1-C5.
- IVC is sound and complete for the class of models satisfying C1-C6.
- IC2 is sound and complete for the class of models satisfying C1-C7.

9 Conclusion

In this paper we studied logics for conditionals built on intuitionistic propositional logic. Following Weiss (2019a,b), we have extended intuitionistic Kripke semantics by equipping models with a family of relations R_X, indexed by propositions, and interpreting a conditional $\varphi > \psi$ as claiming that ψ holds at all the R_φ-successors of the evaluation point. However, we departed slightly from Weiss's proposal. In particular, we restricted to models where the set $R_X[w]$ of successors of a point is upwards-closed with respect to the intuitionistic refinement ordering $\leq$. Technically, this assumption leads to a better-behaved semantics. Conceptually, it can be motivated by the idea that, if a partial state of affairs is considered possible conditionally on φ, then any refinement of it must be considered possible conditionally on φ as well. We have given a general notion of an *intuitionistic conditional logic* and showed how to build, for each such logic, a suitable canonical model.

We then considered seven assumptions on the semantics of conditionals, formulated in terms of constraints on the accessibility relations R_φ. These assumptions are familiar from the literature on conditional logic. Nevertheless, in a couple of cases their specific formulation differs from the one used in the classical case, in order to take into account the upwards-closure requirement on $R_\varphi[w]$ and the fact that, in the intuitionistic setting, worlds are not to be regarded as complete states of affairs, but rather as partial and extendible ones.

We identified conditional axioms corresponding to each condition. Again, these axioms are standard in the conditional logics literature. However, their formulation must be chosen carefully. Two axioms which are equivalent in the classical setting might no longer be equivalent in intuitionistic setting, and it could be that one, but not the other, captures a certain semantic constraint. As an example, Stalnaker's logic C2 is classically axiomatized by extending VC with either of the following two schemata.

- Conditional excluded middle: $(\varphi > \psi) \vee (\varphi > \neg\psi)$
- Conditional determinacy: $(\varphi > \psi \vee \chi) \to (\varphi > \psi) \vee (\varphi > \chi)$

We saw that, in the intuitionistic setting, the conditional determinacy schema gives a sound and complete axiomatization of the logic of the models satisfying Stalnaker's uniqueness assumption C7. By contrast, conditional excluded middle is invalid over this class: even if the satisfaction of conditionals with antecedent φ is determined by looking at the behavior of a single possible world, it need not be the case that this world satisfies either ψ or $\neg\psi$.[14] In fact, as pointed out by Weiss (2019a,b), adding conditional excluded middle to an ICL containing the schema A2 has dramatic consequences: the whole logic becomes classical as a result. Since our goal is to equip intuitionistic logic with a new conditional operator, this is clearly undesirable: whatever our assumptions on $>$, the resulting logic should be a conservative extension of intuitionistic propositional logic.

As another important example, consider the minimal change constraint C4. In the classical case, this constraint is usually characterized by the following axiom (Nute, 1980; Arlo-Costa *et al.*, 2019):

- $((\varphi > \chi) \wedge \neg(\varphi > \neg\psi)) \to (\varphi \wedge \psi > \chi)$

In classical logic, this is obviously equivalent to our schema A4, repeated below:

- $(\varphi > \chi) \to ((\varphi > \neg\psi) \vee (\varphi \wedge \psi > \chi))$

In the intuitionistic setting, however, the two axioms are no longer equivalent. As we saw, the latter axiom fully characterizes the logic which arises from C4. By contrast, the former axiom is sound with respect to models satisfying C4, but it does not yield a complete axiomatization of the associated logic. This illustrates again the fact that in the intuitionistic setting it is crucial to choose the right one among several classically equivalent formulations of a certain principle, i.e., the formulation that correctly captures the intended assumption about the semantics of conditionals.

Similarly, certain connections between the conditional axioms which hold in classical logic no longer hold in the intuitionistic case. For instance, while the strong centering axiom A6 is provable from A2 and A7 classically, we saw that the same is not true in the intuitionistic case. Therefore, strong centering must be taken as an axiom in IC2, the intuitionistic version of Stalnaker's C2.

[14] For a counterexample, consider the model M from the proof of Proposition 8.5. This model satisfies condition C7, since it is finite and linear; therefore, by Theorem 7.2, it validates all instances of conditional determinacy. However, it is immediate to check that neither point in the model satisfies $(p > q) \vee (p > \neg q)$.

Our main result was a general soundness and completeness theorem: any logic axiomatized by a combination of the axioms we identified is sound and complete for the class of models satisfying the corresponding conditions. As a special case, we obtain a completeness result for the intuitionistic counterparts of the classical conditional logics V, VW, VC, and C2.

Summing up, we have seen how intuitionistic logic can be equipped with a Stalnaker-Lewis-style conditional operator, a task which had been left open by Weiss (2019a,b), and we axiomatized the resulting logics. On the way, we observed how some classically equivalent principles of conditional logic come apart in the intuitionistic setting, and how certain intertwined axioms become logically independent.

Several salient directions for further work suggest themselves. First, Lewis (1973) considered already not only the operator $>$, which he denoted as $\Box\!\!\rightarrow$, but also its dual, denoted as $\Diamond\!\!\rightarrow$: whereas $\varphi \Box\!\!\rightarrow \psi$ stands for "if φ were the case, ψ would be the case", $\varphi \Diamond\!\!\rightarrow \psi$ stands for "if φ were the case, ψ might be the case". Just like $\Box\!\!\rightarrow$ can be seen as an antecedent-dependent version of the universal modality $\Box$, $\Diamond\!\!\rightarrow$ can be seen as an antecedent-dependent version of $\Diamond$. In Lewis's theory, these two operators are interdefinable via negation: $\varphi \Diamond\!\!\rightarrow \psi \equiv \neg(\varphi \Box\!\!\rightarrow \neg\psi)$ and $\varphi \Box\!\!\rightarrow \psi \equiv \neg(\varphi \Diamond\!\!\rightarrow \neg\psi)$. Thus, the logic of $\Box\!\!\rightarrow$ uniquely determines the logic of $\Diamond\!\!\rightarrow$, which means that any classical conditional logic given in terms of $\Box\!\!\rightarrow$ extends in a unique way to a logic where the language includes both operators. Things are different in our intuitionistic setting. Just like $\forall$ and $\exists$ are not interdefinable in intuitionistic predicate logic, and $\Box$ and $\Diamond$ are not interdefinable in intuitionistic modal logic, also $\Box\!\!\rightarrow$ and $\Diamond\!\!\rightarrow$ will not be interdefinable in intuitionistic conditional logic either. In order to capture *might*-conditionals in the intuitionistic setting, we need to add $\Diamond\!\!\rightarrow$ to the language as a new primitive operator. It is natural to suppose that the relation between $\Box\!\!\rightarrow$ and $\Diamond\!\!\rightarrow$ will mirror the relation between $\Box$ and $\Diamond$ in intuitionistic modal logic (see Fischer Servi, 1981; Simpson, 1994; Wolter and Zakharyaschev, 1999; Bierman and de Paiva, 2000). In this extended setting, the logical properties of $\Box\!\!\rightarrow$ might not uniquely determine the properties of $\Diamond\!\!\rightarrow$. Therefore, it becomes especially interesting to look at the landscape of intuitionistic conditional logics in a setting where the language comprises both operators.

Second, the properties of a conditional logic depend on two parameters: (i) the underlying propositional logic and (ii) the properties of the operator $>$. The traditional literature fixes (i) to classical logic, and studies the logics that arise from making different assumptions about (ii). In this paper we have done the same, but fixing (i) to intuitionistic logic instead. Between intuitionistic and classical logic there is a variety of interesting intermediate logics, each of which could provide the propositional basis for a family of conditional logics. A generalization of our enterprise in this paper would be to study the general class of *intermediate conditional logics*, where both parameters (i) and (ii) can take a range of different values. For instance, one could consider the behavior of $>$ in the context of the Gödel-Dummett logic LC: this would amount essentially to restricting our semantics to models M where $\leq$ is a linear ordering.

Third, intuitionistic logic can be studied not just by means of Kripke semantics, but also by means of other semantics, including the algebraic semantics based on Heyting algebras, the topological semantics, and the proof-theoretic semantics based on typed λ-calculi. An interesting question is how to extend such alternative semantics in order to interpret the language $\mathcal{L}^{>}$. In the case of algebraic semantics, such an extension has been investigated by Weiss (2019a,b), who provided an algebraic characterization of various weak conditional logics; it would be interesting to extend this work to the stronger logics considered in this paper. Another interesting case is that of proof-theoretic semantics, which formalizes the BHK (Brouwer-Heyting-Kolmogorov) interpretation of connectives. An extension of this semantics to the operator $>$ may cast some light on the following seemingly difficult question: what does it take to constructively prove that, if φ were the case, ψ would be the case?[15]

Fourth, intuitionistic conditional logics have recently been advocated as useful tools in the area of *access control* (Genovese *et al.*, 2014), a subfield of computer science which is concerned with deciding whether an agent should be allowed to perform certain operations. It would be interesting to ask whether the specific logics investigated here are relevant to this kind of applications.

Finally, one may look at the relevance of intuitionistic conditional logics for the analysis of conditionals in natural language. Recently, Ciardelli *et al.* (2018) have provided experimental evidence for two kinds of violations of the predictions of classical conditional logic in natural language. First, antecedents of the form $\neg p \vee \neg q$ and $\neg(p \wedge q)$ do not make the same contribution More precisely, conditionals of the form $(\neg p \vee \neg q) > r$ and $\neg(p \wedge q) > r$ are not in general judged to have the same truth value. This might be taken to count against the principle of replacement of equivalent antecedents. However, one may also retain substitution of equivalent antecedents and base one's theory of conditionals on a logic that invalidates the de Morgan equivalence $\neg p \vee \neg q \equiv \neg(p \wedge q)$. Ciardelli *et al.* (2018) go for the latter option, basing their solution on inquisitive logic (Ciardelli and Roelofsen, 2011). Intuitionistic logic is another natural candidate for such a logic. Second, Ciardelli *et al.* (2018) also provide evidence against the principle $(\neg p > r) \wedge (\neg q > r) \models \neg(p \wedge q) > r$, which is valid in the classical conditional logic $\mathtt{V}$ and its extensions. It is not hard to see that this entailment is not validated by any of the intuitionistic conditional logics considered in this paper. Thus, these logics avoid the problematic empirical predictions of classical conditional logics pointed out by Ciardelli *et al.* (2018). It remains to be seen, however, whether intuitionistic conditional logics also provide a plausible diagnosis for *why* the relevant principles fail in the scenario described by Ciardelli *et al.* (2018).

[15] A good starting point for this enterprise could be the existing work on extending the typed lambda-calculus to intuitionistic modal logics. See Benton *et al.* (1998); Bierman and de Paiva (2000); Alechina *et al.* (2001); Bellin *et al.* (2001); de Paiva and Ritter (2011).

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