

Class 3 : disjunction property, soundness & completeness

Prop (Invariance under generated submodels)

Let M be an i.k.m., w a point in M .

Let M_w be the restriction of M to $R[w]$.

Then $\Vdash_{\Phi} M, w \Vdash \varphi \Leftrightarrow M_w, w \Vdash \varphi$

Proof Immediate, by induction on φ . \square



Cor (Rooted model property)

If $\emptyset \not\models_{IPC} \psi$ there is a model $M = \langle W, R, V \rangle$ and a point $w \in W$ with $R[w] = W$ (we say that w is the root of M) s.t. $M, w \Vdash \varphi$ for all $\varphi \in \emptyset$ but $M, w \not\models \psi$.

Idea: every invalid entailment can be witnessed at the root of an i.k.m.

Proof If $\emptyset \not\models_{IPC} \psi$ then $\exists N, w$ s.t. $N, w \Vdash \varphi$

for all $\varphi \in \emptyset$ but $N, w \not\models \psi$. Let $M := N_w$.

Then w is the root of M and by invariance under gen. submodels we have $M, w \Vdash \varphi \Vdash_{\Phi} \psi$,

$M, w \not\models \psi$. \square

Theorem (Disjunction property)

$$\varphi \vee \psi \in \text{IPC} \Leftrightarrow \varphi \in \text{IPC} \text{ or } \psi \in \text{IPC}$$

NB This fails for classical logic, since, for instance, $p \notin \text{CPC}$, $\neg p \notin \text{CPC}$, $p \vee p \in \text{CPC}$

Proof

\Leftarrow Obvious since $\varphi \models_{\text{IPC}} \varphi \vee \psi$, $\psi \models_{\text{IPC}} \varphi \vee \psi$.

\Rightarrow By contraposition: suppose $\varphi, \psi \notin \text{IPC}$.

Then we have counter-models $M_1, v_1 \Vdash \varphi$ and

$M_2, v_2 \Vdash \psi$. We may assume v_1 is the root of

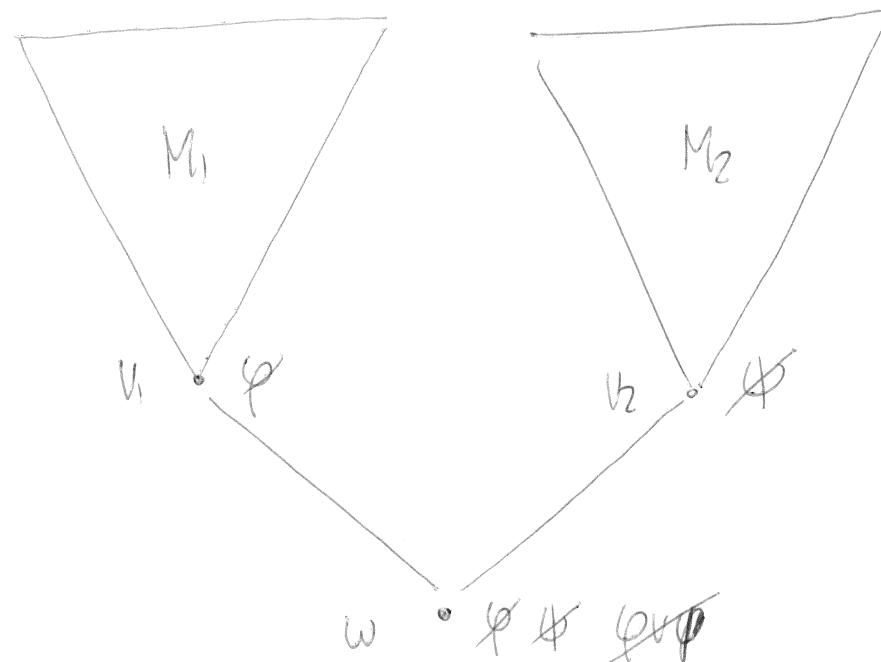
M_1 and ~~M_2~~ w_1, w_2 disjoint. Define a new

model $M = \langle W, R, V \rangle$ as follows:

- $W = W_1 \cup W_2 \cup \{w\}$ with $w \notin W_1 \cup W_2$

- $R = R_1 \cup R_2 \cup \{\langle w, v \rangle \mid v \in W\}$

- $V(p) = V_1(p) \cup V_2(p)$ for all $p \in \Phi$



One can check that R is a partial order and V is persistent, so that M is indeed an i.k.m.

By invariance under gen. submodels, since $M, v_i = M_i$ we have $M, v_i \Vdash \varphi$, $M, v_i \Vdash \psi$. By persistency, it follows $M, w \Vdash \varphi$ and, $M, w \Vdash \psi$. So $M, w \Vdash \varphi \vee \psi$.

Thus $\varphi \vee \psi \notin \text{IPC}$. □

Our main aim for today is to show that the proof-theoretic characterization of IPC and the model-theoretic one coincide:

$$\text{Theor } \bar{\Phi} \vdash_{\text{IPC}} \psi \Leftrightarrow \bar{\Phi} \models_{\text{IPC}} \psi$$

Proof

→ Canonical model construction.

Rest of today's class devoted to it.

← Check soundness of each inference rule.

Ex. (→): Prove that $\bar{\Phi}, \psi \vdash_{\text{IPC}} \chi$ implies $\bar{\Phi} \vdash_{\text{IPC}} \psi \Rightarrow \chi$.

Suppose $\bar{\Phi}, \psi \vdash_{\text{IPC}} \chi$. Take any M, w s.t. $M, w \Vdash \bar{\Phi}$.

Take any $v \in R(w)$ with $M, v \Vdash \psi$. By persistency

also $M, v \Vdash \bar{\Phi}$, and since $\bar{\Phi}, \psi \vdash_{\text{IPC}} \chi$, $M, v \Vdash \chi$.

This proves that $M, w \Vdash \psi \Rightarrow \chi$. So, $\bar{\Phi} \vdash_{\text{IPC}} \psi \Rightarrow \chi$.

Canonical model construction

Def A set $P \subseteq \mathcal{L}_0$ is an IPC-theory if $\forall \varphi: P \vdash_{\text{IPC}} \varphi \Rightarrow \varphi \in P$.

Def Let P be an IPC-theory. We say that:

- P is consistent if $\perp \notin P$
- P has the v-property if $\forall \varphi, \psi: \varphi \vee \psi \in P \Rightarrow \varphi \in P \text{ or } \psi \in P$,

Def The canonical model for IPC is M^c
= $\langle W^c, R^c, V^c \rangle$ where:

- W^c is the set of consistent IPC-theories with the v-property
- $R^c = \subseteq$ (i.e., $P R^c P' \Leftrightarrow P \subseteq P'$)
- $\forall p \in P: \Gamma^c \in V^c(p) \Leftrightarrow p \in \Gamma$

Rem M^c is an I.K.m.:

- \subseteq is a partial order
- V^c is persistent, since

$$\begin{aligned}\Gamma \in V^c(p) \text{ & } \Gamma \vdash \Gamma' &\Rightarrow p \in \Gamma \text{ & } \Gamma \subseteq \Gamma' \\ &\Rightarrow p \in \Gamma' \\ &\Rightarrow \Gamma' \in V^c(p)\end{aligned}$$

Lemma (Lindenbaum-type)

$$\Phi \not\vdash_{IPC} \psi \Rightarrow \exists \Gamma \in W: \Phi \subseteq \Gamma \text{ but } \psi \notin \Gamma.$$

Proof Suppose $\Phi \not\vdash_{IPC} \psi$. Enumerate $L_p = \{p_0, p_1, \dots\}$.

$$\begin{aligned}- P_0 &:= \Phi \\ - P_{n+1} &= \begin{cases} P_n \cup \{p_n\} & \text{if } P_n \cup \{p_n\} \not\vdash_{IPC} \psi \\ P_n & \text{otherwise} \end{cases}\end{aligned}$$

$$P = \bigcup_{n=1}^{\infty} P_n$$

Obviously $\Phi \subseteq P$ and $\psi \notin P$.

To show: $P \in W$, i.e., P is a consistent theory with the V-property.

We first show a preliminary claim.

Claim: $P \not\vdash \psi$.

Suppose $P \vdash \psi$. Then there are $f_1, \dots, f_n \in P$ s.t. $f_1 \dots f_n \vdash \psi$. Let $m \in \mathbb{N}$ be such that $f_1, \dots, f_n \in P_m$. Then $P_m \vdash \psi$. But this is impossible: an immediate induction on m shows that $P_m \not\vdash \psi$ for all $m \in \mathbb{N}$.

Now we can show that P is an IPC theory.

Suppose $P \vdash X$. Then $P \cup \{X\} \vdash \psi$, otherwise $P \vdash \psi$. Let $X = p_m$. Since $P_m \subseteq P$, $P_m \cup \{X\} \vdash \psi$, so by definition $P_{m+1} = P_m \cup \{X\}$, and then $X \in P$.

Also, P is consistent: this is obvious since $P \not\vdash \psi$, whereas if $1 \in P$ we would have $P \vdash X$ for all X .

Finally, we need to show that Γ has the V-property. We show that if $\varphi_1, \varphi_m \notin \Gamma$, then $\varphi_1 \vee \varphi_m \notin \Gamma$. So, suppose $\varphi_1, \varphi_m \notin \Gamma$. Then:

$$\left. \begin{array}{l} \Gamma \cup \{\varphi_1\} \vdash \psi \xrightarrow{\Gamma \subseteq \Gamma'} \Gamma' \cup \{\varphi_1\} \vdash \psi \\ \Gamma \cup \{\varphi_m\} \vdash \psi \xrightarrow{\Gamma \subseteq \Gamma'} \Gamma' \cup \{\varphi_m\} \vdash \psi \end{array} \right\} \stackrel{\text{re}}{\Rightarrow} \Gamma \cup \{\varphi_1 \vee \varphi_m\} \vdash \psi$$

Since $\Gamma \nvdash \psi$, it follows $\varphi_1 \vee \varphi_m \notin \Gamma$. \square

Now we are ready to prove that the canonical model gives us the desired bridge between syntax and semantics.

Lemma (Truth-lemma) $M^c, \Gamma \vdash \varphi \Leftrightarrow \varphi \in \Gamma$

Proof By induction on φ .

- Atoms: by definition of V^c
- Inductive steps for \wedge, \vee, \perp : straightforward.
- Inductive step for \rightarrow : suppose $\varphi = \psi \rightarrow \chi$.

\Leftarrow . Suppose $\psi \rightarrow \chi \in \Gamma$. To show: $M^c, \Gamma \vdash \psi \rightarrow \chi$. Take $\Gamma' \supseteq \Gamma$ with $M^c, \Gamma' \vdash \psi$. By IH, $\psi \in \Gamma'$. Since $\Gamma \subseteq \Gamma'$, $\psi \rightarrow \chi \in \Gamma'$. Since $\psi, \psi \rightarrow \chi \in \Gamma'$ and Γ' is an IPC-theory, $\chi \in \Gamma'$. By IH, $M^c, \Gamma' \vdash \chi$. This shows that, indeed, $M^c, \Gamma \vdash \psi \rightarrow \chi$.

\Rightarrow . Suppose $\psi \rightarrow \chi \notin \Gamma$.

Since Γ is an IPC-theory, $\Gamma \nvdash \psi \rightarrow \chi$. So by (ii), $\Gamma, \psi \nvdash \chi$.

By Lindenbaum-type lemma, there is $\Gamma' \in W^c$ s.t. $\Gamma \cup \{\psi\} \subseteq \Gamma'$ but $\chi \notin \Gamma'$. So we have:

$$\left. \begin{array}{l} \Gamma \subseteq \Gamma' \xrightarrow{\text{IPC}} \Gamma \vdash \psi \\ \psi \in \Gamma' \xrightarrow{\text{IH}} M^c, \Gamma' \vdash \psi \\ \chi \notin \Gamma' \xrightarrow{\text{IH}} M^c, \Gamma' \nvdash \chi \end{array} \right\} \text{So there is a successor of } \Gamma \text{ which forces } \psi \text{ but not } \chi, \text{ which leads to } M^c, \Gamma \vdash \psi \rightarrow \chi.$$

\square

Proof of Completeness

Suppose $\Phi \nvdash_{\text{PC}} \Psi$.

By Lindenbaum lemma, $\exists \Gamma \in W^e$ s.t. $\Phi \subseteq \Gamma$, $\Psi \notin \Gamma$.

By Truth lemma, $M^c \Gamma \Vdash \varphi$ for all $\varphi \in \Phi$, $M^c \Gamma \not\Vdash \Psi$.

Therefore, $\Phi \nvdash_{\text{PC}} \Psi$. □