Intuitionistic Logic

Exercise sheet 7

November 25, 2019

Exercise 1. [Topological semantics]
The soundness and completeness of IPC for the topological semantics can be extended from validity to entailments as follows:

$$\varphi \vdash_{\mathsf{IPC}} \psi \iff \text{for all topological models } T : |\varphi|_T \subseteq |\psi|_T$$

Use the topological semantics to show that $\neg p \vee \neg q \vdash_{\mathsf{IPC}} \neg(p \wedge q)$ but $\neg(p \wedge q) \nvdash_{\mathsf{IPC}} \neg p \vee \neg q$.

Exercise 2. [Topological semantics]
Given a set $X \subseteq W$ within a topological space $S = \langle W, \tau \rangle$, its closure X^{cl} is the least closed set which contains X. Such a set always exists, and can be characterized in the following ways:

$$X^{cl} = \bigcap \{Y \subseteq W \mid Y \text{ is closed and } X \subseteq Y\} = \overline{(\overline{X})^{int}}$$

The set X is said to be *dense* if $X^{cl} = W$. Intuitively, this captures the fact that any point in W has points in X which are indefinitely close to it: for instance, the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$. Show that the topological semantic provides the following characterization of classical logic:

$$\varphi \in \mathsf{CPC} \iff |\varphi|_T \text{ is dense for every topological model } T$$

Hint: use Glivenko's theorem.

Exercise 3. [Heyting algebras]

- Draw all Heyting algebras with five elements.
- Use these algebras to give HA-countermodels to the validity of $p \vee \neg p$, $\neg\neg p \to p$, and $(\neg q \to \neg p) \to (p \to q)$.

Exercise 4. [Heyting algebras]
We have defined HA's as posets with extremal elements, and in which every two elements a, b have a meet $a \wedge b$ (greatest lower bound), a join $a \vee b$ (least upper bound), and an implication $a \rightarrow b$ (an element s.t. $a \wedge c \leq b \iff c \leq a \rightarrow b$).

Sometimes (e.g., in the lecture notes that you find on the course webpage) HA's are required to satisfy in addition the following distributive laws:

- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Show that the validity of these laws actually follows from the HA properties.

Hint: use the properties of implication to prove the first law; then use the first law to prove the second.